

# Generalized Lyapunov Exponent and Transmission Statistics in One-dimensional Gaussian Correlated Potentials

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Distribution of the transmission coefficient  $T$  of a long system with a correlated Gaussian disorder is studied analytically and numerically in terms of the generalized Lyapunov exponent (LE) and the cumulants of  $\ln T$ . The effect of the disorder correlations on these quantities is considered in weak, moderate and strong disorder for different models of correlation. Scaling relations between the cumulants of  $\ln T$  are obtained. The cumulants are treated analytically within the semiclassical approximation in strong disorder, and numerically for an arbitrary strength of the disorder. A small correlation scale approximation is developed for calculation of the generalized LE in a general correlated disorder. An essential effect of the disorder correlations on the transmission statistics is found. In particular, obtained relations between the cumulants and between them and the generalized LE show that, beyond weak disorder, transmission fluctuations and deviation of their distribution from the log-normal form (in a long but finite system) are greatly enhanced due to the disorder correlations. Parametric dependence of these effects upon the correlation scale is presented.

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## I. INTRODUCTION

Properties of stochastic systems can be significantly affected by the degree of correlation of the external noise term (see, e.g., Ref. [1] and references therein). As an example of such phenomenon, we study statistical properties of Anderson localization [2] in a linear one-dimensional model

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = \epsilon\psi, \quad (1)$$

with correlated disorder potential  $V(x)$ . With a suitable change of notation, equation (1) can describe a stationary problem either for a quantum particle at energy  $\epsilon$ , or for classical scalar electromagnetic or acoustic waves. Equation (1) appears also in many other fields of physics. For example, with  $x$  considered time, Eq. (1) represents a random frequency oscillator, which is a simple paradigm of a stochastic dynamical system (see Ref. [3] for a more detailed discussion of this and other applications). In all these instances, an important quantity is the Lyapunov exponent (LE)  $\gamma = \lim_{x \rightarrow \infty} \tilde{\gamma}(x)$ , where  $\tilde{\gamma}(x) = \frac{1}{2x} \ln \left( |\psi|^2 + |\psi'|^2 \right)$  and  $\psi$  is a solution of an initial value problem. LE is a non-random quantity independent of the specific realization of disorder  $V(x)$  [4]. On the contrary, "local" LE  $\tilde{\gamma}(x)$  is random and should be described statistically by distribution  $P(\tilde{\gamma}; x)$ . Distribution  $P(\tilde{\gamma}; x)$  can be studied in terms of its cumulants (e.g. Ref. [5]) or using the generalized Lyapunov exponents  $\Lambda^{(q)} = \lim_{x \rightarrow \infty} \frac{1}{2qx} \ln \left\langle \left( |\psi|^2 + |\psi'|^2 \right)^{q/2} \right\rangle$ , where  $\langle \dots \rangle$  denotes average over the disorder realizations (e.g., Refs. [3],[6]).

LEs  $\gamma$  and  $\tilde{\gamma}(x)$  are intimately related to the localization properties of one-dimensional disordered systems. The latter was studied extensively and it is rigorously established that almost all eigenstates in one-dimensional

(1D) disordered systems are exponentially localized under rather general conditions [7],[8],[9] (see also [4],[10] and references therein). The inverse localization length of the eigenstates, as well as the asymptotic decay rate  $\lim_{L \rightarrow \infty} (2L)^{-1} \ln T^{-1}(L)$  of the transmission coefficient  $T$  of the system of length  $L$ , are equal [4] to the LE  $\gamma$ . More generally, statistical properties of the quantity  $(2L)^{-1} \ln T^{-1}(L)$  and of the introduced above local LE  $\tilde{\gamma}(x=L)$  become asymptotically equivalent for large  $L$  (see Sec. II). Therefore, in what follows, we discuss the problem in terms of the transmission coefficient  $T$ , while the results apply in a broader context of the local LE  $\tilde{\gamma}$ .

Transmission coefficient is a basic characteristics of the wave transport through the non-uniform media. In the electronic systems,  $T$  is related to the dimensionless conductance by the Landauer formula [11]. Measurement of the transmission coefficients is one of the ways to investigate different aspects of Anderson localization experimentally. Some indications of Anderson localization were observed with light [12], microwaves [13], cold atoms [14] and ultrasound [15]. In most of these experiments, the correlation scale of the randomness is comparable to the scattered wavelength and should be taken into account to obtain a full quantitative description. In the experiments in Ref. [14], localization of cold atoms was found by observing localized density profiles, rather than transmission. Recently, possible experiments on cold atom transmission through the disordered optical potentials were discussed [16]. It is worth noting that in the actively developing field of cold atoms, one of the ways to introduce disorder is by using the laser speckle intensity patterns, which are highly correlated [17]. On the theoretical side, there is an increasing interest in the disordered systems with correlated potentials. In particular, in 1D systems, disorder correlations were found to have strong effect both on the localization length [4],[18],[19],[20],[21], on the generalized LE [22],[23] and on the transmission statistics [24],[25] and, in special cases, even lead to the

appearance of extended states [26].

The main subject of the present work, is the effect of disorder correlations on the transmission statistics in the asymptotic limit of large  $L$ . Transmission coefficient of a long disordered system exhibits large sample-to-sample fluctuations and is not a self-averaging quantity. General properties of the transmission distribution have been considered in the past using the composition rule for the transmission of a one-dimensional chain of statistically identical and independent (or only weakly dependent) random scatterers [27],[28]. It was shown that a convenient variable to deal with is  $\ln T(L)$ , since it can be represented as a sum of independent (or weakly dependent) random variables, for which the conditions of the Central Limit theorem, or its modification for weakly correlated variables, are valid. Then, by additivity of cumulants of independent variables, it follows that all cumulants of the  $\ln T$ -distribution,  $P(\ln T; L)$ , grow at most linearly in the system length  $L$ :

$$\left\langle\left\langle\left[\ln T^{-1/2}(L)\right]^n\right\rangle\right\rangle = c_n L + o(L), \quad n = 1, 2, \dots \quad (2)$$

Here  $\langle\langle X^n \rangle\rangle$  denotes  $n$ th-order cumulant of the quantity  $X$ , and the asymptotic cumulant coefficients  $c_n$  are constants, which depend on the microscopic properties of the system. The above result applies to systems with both uncorrelated and correlated random potentials, provided the considered system can be divided into blocks, which are much longer than the disorder correlation scale and can be treated as nearly independent scatterers.

Thus, the general form of the large- $L$  limit of the transmission distribution is well understood, and the remaining questions are about the dependence of the cumulant coefficients  $c_n$  on the parameters of the problem. According to Eq. (2), transmission statistics can be discussed at the following three levels of "resolution". First, there exists a self-averaging quantity  $(2L)^{-1} \ln T^{-1}$ , which has a non-random limit

$$\lim_{L \rightarrow \infty} (2L)^{-1} \ln T^{-1} = c_1 \equiv \gamma. \quad (3)$$

At this level, all the information about the transmission fluctuations is lost in the asymptotic limit  $L \rightarrow \infty$ . Next, one can define a variable  $(\ln T - \langle \ln T \rangle) / (4c_2 L)^{1/2}$ , where  $\langle \ln T \rangle = 2c_1 L$ , for which the Central Limit theorem holds, and whose cumulants of the order  $n \geq 3$  vanish in the asymptotic limit  $L \rightarrow \infty$  as  $L^{-(n-2)/2}$ . Therefore, this variable has a limiting Gaussian distribution with zero mean and unit variance. Limiting probability distributions and their relation to universality and scaling have been discussed in the past [28],[29]. In the present context of a 1D problem, only two model-dependent parameters,  $c_1$  and  $c_2$ , are "remembered" in the limiting distribution of  $(\ln T - \langle \ln T \rangle) / (4c_2 L)^{1/2}$ . This fact is referred to as "two-parameter scaling" [28]. In special conditions, namely when the combined scatterers are weak and their reflection phases are distributed uniformly, the

composition rule yields  $c_1 = c_2$  up to weak disorder corrections, which is a manifestation of single parameter scaling (SPS) [27],[28].

Finally, for large but finite  $L$ , one can study the non-Gaussian corrections to the limiting Gaussian distribution, which are characterized by the asymptotic cumulant coefficients  $c_n$ ,  $n \geq 3$ . The latter are responsible for the extreme fluctuations of the transmission coefficient. These large deviations can also be studied in terms of the generalized LEs, which describe asymptotic growth rates of the moments of the inverse transmission coefficient  $T^{-1}$  and can be expressed as a sum over  $c_n$ 's [3].

Under conditions of weak scattering and phase randomization, in addition to the SPS relation  $c_1 = c_2$ , it was found in some models of disorder that cumulant ratios  $c_n/c_1$ ,  $n \geq 3$ , vanish up to the weak disorder corrections [5],[30]. Vanishing of higher cumulants is also indicated by the coincidence of LE  $\gamma$  and the second order generalized LE  $\Lambda$  in the lowest order of the weak disorder expansion, which is valid for a wide variety of 1D disordered models [3],[4],[19],[22]. Thus, quite generally, cumulant coefficients  $c_n$  satisfy

$$c_2/c_1 = 1, \quad c_n/c_1 = 0, \quad n \geq 3, \quad (4)$$

which holds up to the weak disorder corrections.

Considerable analytical and numerical effort was devoted to verification of SPS and determining limits of its validity for various models of disorder [5],[28],[30],[31],[32],[33],[34]. Usually, SPS holds for sufficiently weak disorder, when the localization length is much larger than other scales in the problem (see also discussion in Ref. [32]).

Deviations from relations (4) were studied for uncorrelated disorder [3],[5],[35] as well as for some specific models of correlated disorder potentials [24],[25]. In particular, for the continuous white noise model, it was shown [5] that  $c_2/c_1 = 1$  for large positive energy  $\epsilon$  (weak disorder), becomes slightly different from unity near  $\epsilon = 0$  and vanishes for large negative  $\epsilon$ , while the "non-Gaussian" ratios  $c_{n \geq 3}/c_1$  have numerical values of about a fraction of unity near  $\epsilon = 0$ , and vanish for large negative or positive  $\epsilon$  (energy scale is determined by strength of disorder). Thus, in the white noise model, unless  $\epsilon$  is sufficiently large and negative, the relative width of the Gaussian bulk of the  $\ln T$ -distribution depends weakly on the disorder strength, while finite higher cumulants indicate that some non-Gaussian corrections to the distribution tails develop in strong disorder. The latter was also observed in Ref. [3], where distribution of local Lyapunov exponents was studied in terms of the generalized LEs for a continuous  $\delta$ -correlated disorder.

Presence of disorder correlations can significantly enhance deviation from relations (4). For instance, one finds  $|c_2/c_1 - 1| \sim 1$  near certain energies of the tight-binding model with a weak short-range correlated disorder [24]. Another example of the effect of correlations is scaling relation  $c_2/c_1 \sim c_1 R_c$ , obtained in Ref. [25] for strong exponentially correlated dichotomous disorder

with a correlation scale  $R_c$ . This scaling, which replaces Eq. (4), is valid when  $c_1 R_c \gg 1$ , which means  $c_2/c_1 \gg 1$ , so that Gaussian bulk of the distribution  $P(\ln T; L)$  becomes much broader than in weak or white noise disorder.

These special examples suggest that disorder correlations can have a significant effect on the transmission statistics. One can distinguish two limiting regimes. In weak disorder, which can be treated perturbatively, LE depends strongly on the correlation scale  $R_c$  [4],[18],[20],[21]. However, it usually remains the only parameter of the transmission distribution, and relations (4) stay valid [30],[33],[34]. Another special regime, which can be called a semiclassical regime of strong disorder, occurs when typical random potential barriers are higher than energy  $\epsilon$  and sufficiently broad, so that tunneling probability already through a single random barrier becomes small. In this case localization is dominated by the under-barrier tunneling, and one can generalize the semiclassical approach of Ref. [25] to study the distribution of  $\ln T(L)$  in terms of the statistics of disorder excursions above the level  $V(x) = \epsilon$ . In the present work we use this approximation for the strong disorder regime to show that  $c_n/c_1 \sim (c_1 R_c)^{n-1}$ .

The main focus of the present research is, however, on the effect of disorder correlations on the transmission distribution at the transition between these two limiting regimes. We consider model (1) with Gaussian disorder  $V(x)$ , whose correlation function is arbitrary, except for the assumption that it is parametrized with a finite correlation scale  $R_c$ . The transmission statistics is studied in terms of the dimensionless ratios  $c_2/c_1$ ,  $c_3/c_1$  and  $\Lambda/c_1$ . Quantity  $\Lambda/c_1$  characterizes extreme fluctuations of the inverse transmission coefficient  $T^{-1}$ , since it is related to the ratio of the average to the typical values of  $T^{-1}$ . In addition, it can be used to show violation of relations (4), because  $\Lambda/c_1 = 1$  when these relations hold [3],[22].

To calculate  $\Lambda$ , which can not be treated exactly in a general case of correlated disorder, we develop a small- $R_c$  approximation [36]. A simple regularization is proposed to extend this method to the power law correlations, for which a standard approximation turns out to be inapplicable. Numerical simulations are used to verify the analytical approximation for  $\Lambda$  and to calculate cumulant coefficients  $c_{1,2,3}$  for different types of correlations and for different values of  $R_c$  (and  $\epsilon > 0$ ). As expected, the obtained results for  $c_2/c_1$ ,  $c_3/c_1$  and  $\Lambda/c_1$  show that SPS and Eq. (4) hold in the weak disorder limit. Beyond this regime, we find that disorder correlations strongly enhance deviations from relations (4), as compared to the case of white noise, and lead to increase of both typical and extreme fluctuations of transmission. Dependence of these effects on the parameters  $kR_c$  and  $c_1 R_c$  is discussed ( $k = \sqrt{\epsilon}$  is the wavenumber). In general, effect of disorder correlations on the transmission statistics becomes important when  $c_1 R_c$ , i.e. the ratio between the correlation and the localization lengths, is comparable to or greater than unity.

The outline of the paper is as follows. In Sec. II we

define the model in question and discuss in some more detail the standard and the generalized Lyapunov exponents. The small- $R_c$  approximation for the generalized LE  $\Lambda$  is developed in Sec. III. First, using the analogy of Eq. (1) with the Langevin equation, equations for the second moments of  $\psi(x)$  are obtained by means of the Furutsu-Novikov formula. This leads to an infinite hierarchy of the integro-differential equations for the second moment of the wave function and its functional derivatives [1],[36]. Then, the small- $R_c$  closure approximation is applied and the result is compared to numerical simulations. In Sec. IV we obtain scaling relations for  $c_n/c_1$  in the semiclassical regime of strong disorder. In Sec. V analytical and numerical results for  $\Lambda$  and  $c_n$  are used to discuss properties of the asymptotic distribution of the transmission coefficient and deviation from relations (4). The results are summarized in Sec. VI. Known results on the generalized LE for the  $\delta$ -correlated disorder and on the Born approximation are presented in appendices A and B respectively. Method of numerical simulation is explained in Appendix C. Some auxiliary formulae for the semiclassical treatment of strong disorder are presented in Appendix D.

## II. BASIC EQUATIONS

We consider one-dimensional Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = \epsilon\psi, \quad (5)$$

where  $\epsilon$  is energy and  $V(x)$  is zero-mean Gaussian disorder potential. The latter is completely described by its two-point correlation function  $C_2(x - x') \equiv \langle V(x)V(x') \rangle$ , written as

$$C_2(x - x') = V_0^2 \Gamma\left(\frac{x - x'}{R_c}\right), \quad (6)$$

where  $V_0^2$  and  $R_c$  are disorder variance and correlation length, and the dimensionless function  $\Gamma(x)$  decays on the scale of unity and is chosen so that

$$\Gamma(0) = 1, \quad \int_0^{+\infty} \Gamma(x) dx = 1. \quad (7)$$

The white noise model is obtained by taking the limit  $R_c \rightarrow 0$ , while keeping  $R_c V_0^2$  constant:

$$\lim_{R_c \rightarrow 0} C_2(x - x') = g\delta(x - x'), \quad (8)$$

where we have introduced disorder intensity parameter

$$g \equiv \int_{-\infty}^{+\infty} C_2(x) dx = 2R_c V_0^2. \quad (9)$$

The transmission properties of a one-dimensional system of finite length  $L$  are described by transfer matrix [37]

$$\frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\phi_t} & -\sqrt{1-T}e^{-i(\phi_r-\phi_t)} \\ -\sqrt{1-T}e^{i(\phi_r-\phi_t)} & e^{-i\phi_t} \end{bmatrix}, \quad (10)$$

which relates amplitudes of the incident and the outgoing waves (a time reversal invariance is implied). Here  $\phi_r$  and  $\phi_t$  are reflection and transmission phases respectively. In the following sections, generalized LE  $\Lambda$  and asymptotic cumulant coefficients  $c_n$  are studied in terms of the amplitude  $A^2(x) = |\psi(x)|^2 + |\psi'(x)|^2$ , where  $\psi(x)$  is a solution of (5) satisfying some generic initial conditions, e.g.  $\psi(0) = 1$ ,  $\psi'(0) = 0$ . The obtained results are valid also for the transmission coefficient  $T$ , since in the asymptotic limit  $L \rightarrow \infty$  it becomes statistically equivalent to the amplitude  $A(x)$ . This is because  $A(x)$  is expressed in terms of the transfer matrix (10), which asymptotically factorizes into the product of large factor  $T^{-1}$  and statistically independent of it matrix of the order of unity. Such factorization takes place, since for  $L$  much larger than the localization length, phases  $\phi_r$  and  $\phi_t$  become statistically (almost [38]) independent of  $T$ , while coefficient  $T$  becomes exponentially small in most realizations of disorder. Thus, for large  $L$ , one can write

$$\ln A^2(L) = \ln T^{-1}(L) + O(L^0), \quad (11)$$

where the term  $O(L^0)$ , which absorbs phases  $\phi_{r,t}$  and the initial conditions for  $\psi(x)$ , is statistically independent of the first one, and its cumulants are of the order of unity. Therefore, using additivity of cumulants of independent variables, one has (cf. Eq.(2))

$$c_n \equiv \lim_{L \rightarrow \infty} \frac{\langle \ln^n T^{-1} \rangle}{2^n L} = \lim_{L \rightarrow \infty} \frac{\langle \ln^n A^2 \rangle}{2^n L}. \quad (12)$$

Thus, for large  $L$ , inverse transmission coefficient  $T^{-1}$  is statistically equivalent to the amplitude  $A^2(x=L)$ , and, thus, to the local LE  $\tilde{\gamma}(x=L)$ , defined in the introduction. In particular, LE  $\gamma$  coincides with  $c_1$ , and in the following we will use the latter notation.

As already mentioned in the introduction, cumulant coefficients  $c_n$  describe asymptotic form of the distribution  $P(\ln T; L)$ . Introducing the dimensionless length  $l \equiv c_1 L$  measured in units of the localization length  $c_1^{-1}$ , cumulants of  $\ln T$  can be rewritten as (cf. Eq. (2))

$$\langle \ln^n T^{-1/2}(l) \rangle = \frac{c_n}{c_1} l + O(l^0). \quad (13)$$

Therefore, it is convenient to discuss transmission distributions in terms of the dimensionless ratios  $c_n/c_1$ , since meaningful comparison of transmission fluctuations in different models of disorder should be done for the same dimensionless system length  $l$ . Then, for instance, relative fluctuation of  $\ln T$  is expressed as  $\langle (\ln^2 T) \rangle / \langle \ln T \rangle^2 = (c_2/c_1)/l$ .

Another quantity which characterizes transmission distribution is second order generalized LE  $\Lambda$  defined by [6]

$$\Lambda = \lim_{L \rightarrow \infty} \frac{1}{4L} \ln \langle A^2(L) \rangle = \lim_{L \rightarrow \infty} \frac{1}{4L} \ln \langle T^{-1}(L) \rangle, \quad (14)$$

Since  $\ln \langle A^q(x) \rangle = \ln \langle e^{q \ln A(x)} \rangle$  is a cumulant generating function of the distribution  $P(\ln A; x)$ , the generalized LE is related to all cumulant coefficients  $c_n$  by the

expression

$$\Lambda = \frac{1}{4} \sum_{n=1}^{\infty} \frac{2^n}{n!} c_n. \quad (15)$$

For large  $L$ , value of  $\langle T^{-1}(L) \rangle$  is dominated by the long tail of the  $T^{-1}$ -distribution, which becomes highly skewed and heavy-tailed. Therefore, rather than representing typical values of  $T^{-1}$ , generalized LE  $\Lambda$  provides useful complementary information about the low- $T$  tail of its distribution. To this end, it is convenient to define the following quantity

$$\rho \equiv \frac{\Lambda}{c_1} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{2^n}{n!} \frac{c_n}{c_1}, \quad (16)$$

whose meaning is twofold. On the one hand, deviation of the value of  $\rho$  from unity gives an integral measure of deviation from the universal weak disorder relations (4), since  $\rho = 1$  when these relations hold (cf. Ref. [3]). On the other hand,  $\rho$  can be used to characterizes the extreme relative fluctuations of the inverse transmission coefficient  $T^{-1}$  in terms of the ratio between the mean and the typical values of  $T^{-1}$ :

$$\frac{\langle T^{-1} \rangle}{\exp \langle \ln T^{-1} \rangle} = e^{(2\rho-1)2l}. \quad (17)$$

### III. GENERALIZED LYAPUNOV EXPONENT $\Lambda$

#### A. Equation of motion for moments

The linearity of the Schrödinger equation (5) allows one to obtain closed-form equations for the  $2n$ -order products  $[\psi(x)]^k [\psi'(x)]^l$ , where  $k+l=2n$ ,  $k, l=0, 1, 2, \dots$ . To this end, we rewrite Eq. (5) in the form of the Langevin equation. The  $x$  coordinate is considered as a formal time on the half axis  $x \equiv t \in [0, \infty)$  and the dynamical variables  $u(t) \equiv \psi(x)$ ,  $v(t) \equiv \psi'(x)$  are introduced. In these variables the Langevin equation reads

$$\partial_t u = v, \quad \partial_t v = [V(t) - \epsilon]u, \quad (18)$$

where  $V(t)$  is now the correlated noise. We need to consider an initial value problem for the second order moments  $\langle |u|^2 \rangle$ ,  $\sqrt{2^{-1}} \langle u^* v + uv^* \rangle$ ,  $\langle |v|^2 \rangle$ , whose asymptotic exponential growth rate gives the generalized LE  $\Lambda$ , Eq. (14). Introducing vector

$$\mathbf{Y} = (|u|^2, \sqrt{2^{-1}} [u^* v + uv^*], |v|^2)^T, \quad (19)$$

and using Eq. (18), one obtains

$$\partial_t \mathbf{Y} = (\mathcal{C} + V(t) \mathcal{D}) \mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0, \quad (20)$$

where

$$\mathcal{C} = \sqrt{2} \begin{bmatrix} 0 & 1 & 0 \\ -\epsilon & 0 & 1 \\ 0 & -\epsilon & 0 \end{bmatrix}; \quad \mathcal{D} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (21)$$

and  $\mathbf{Y}_0$  is an initial condition. To obtain equation for  $\langle \mathbf{Y}(t) \rangle$ , Eq. (20) is averaged over the disorder, which introduces correlator  $\langle V(t) \mathbf{Y}(t) \rangle$ :

$$\partial_t \langle \mathbf{Y} \rangle = \mathcal{C} \langle \mathbf{Y} \rangle + \mathcal{D} \langle V(t) \mathbf{Y} \rangle, \quad \langle \mathbf{Y}(0) \rangle = \mathbf{Y}_0. \quad (22)$$

Applying the Furutsu-Novikov formula [1],[36] yields

$$\partial_t \langle \mathbf{Y} \rangle = \mathcal{C} \langle \mathbf{Y} \rangle + \int_0^t C_2(t-\tau) \mathcal{D} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle d\tau, \quad (23)$$

where  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  is a functional derivative of  $\mathbf{Y}(t)$  with respect to  $V(\tau)$  and  $C_2(t)$  is the disorder correlation function (6). Since equation (23) contains a new quantity  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$ , it is not closed with respect to  $\langle \mathbf{Y} \rangle$ . An important exception is the case of the uncorrelated disorder, which is considered in Appendix A, and will be used in the forthcoming analysis. In general case, to proceed further, an additional equation for  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  is required, which is obtained by differentiating Eq. (20) functionally with respect to  $V(\tau)$ , and using the differentiation property  $\frac{\delta}{\delta V(\tau)} \frac{\partial}{\partial t} \mathbf{Y}(t) = \frac{\partial}{\partial t} \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  [36]. This yields

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} &= \mathcal{D} \mathbf{Y}(\tau) \delta(t-\tau) + \\ &+ \Theta(t-\tau) [\mathcal{C} + V(t) \mathcal{D}] \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}, \quad \frac{\delta \mathbf{Y}(t < \tau)}{\delta V(\tau)} = 0, \end{aligned} \quad (24)$$

where  $\Theta(t-\tau)$  is the Heaviside step function. For  $t > \tau$ , Eq. (24) can be rewritten as

$$\frac{\partial}{\partial t} \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} = [\mathcal{C} + V(t) \mathcal{D}] \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}, \quad \frac{\delta \mathbf{Y}(\tau^+)}{\delta V(\tau)} = \mathcal{D} \mathbf{Y}(\tau). \quad (25)$$

The causality property  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau > t)} = 0$  was used in derivation of Eq. (24). It follows from the fact that  $\mathbf{Y}(t)$  obeys first order differential equation (20) with an initial condition at  $t = 0$  and, thus, is independent of  $V(\tau)$  at the time  $\tau$  later than  $t$ . Then, integration of the differential equation (24) over  $t$  from some  $t_1 < \tau$  up to  $\tau^+$  yields the initial condition in Eq. (25).

Note that functional derivative  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  in Eq. (25) and vector  $\mathbf{Y}(t)$  in Eq. (20) obey the same differential equation, but with different initial condition. Due to this similarity, moments  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$  and  $\langle \mathbf{Y}(t) \rangle$  grow with the same asymptotic rate for large  $(t-\tau)$  and  $t$  respectively. The difference between the two quantities is that the initial condition for  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  depends on  $\mathbf{Y}(\tau)$  and, therefore, it is correlated to  $V(t)$ .

Averaging Eq. (25) over the disorder and using the Furutsu-Novikov formula, one obtains the following equation for the moment  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$ :

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle &= \mathcal{C} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle + \\ &+ \int_0^t C_2(t-\tau') \mathcal{D} \left\langle \frac{\delta^2 \mathbf{Y}(t)}{\delta V(\tau') \delta V(\tau)} \right\rangle d\tau' \end{aligned} \quad (26)$$

with the initial condition

$$\left\langle \frac{\delta \mathbf{Y}(t = \tau^+)}{\delta V(\tau)} \right\rangle = \mathcal{D} \langle \mathbf{Y}(\tau) \rangle. \quad (27)$$

Thus, Eq. (26) is again not closed with respect to  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$ , because it contains an additional quantity  $\left\langle \frac{\delta^2 \mathbf{Y}(t)}{\delta V(\tau') \delta V(\tau)} \right\rangle$ . Then, the above procedure can be iterated repeatedly to obtain an infinite hierarchy of coupled differential equations involving moments of higher functional derivatives of  $\mathbf{Y}$  with respect to  $V(t)$  [1],[36].

## B. Small- $R_c$ decoupling approximation

To make progress with the infinite hierarchy of coupled differential equation, the first two of which are equations (23) and (26), one usually resorts to some closure approximation [1],[36]. In this work we apply a small- $R_c$  approximation to close the second equation (26) with respect to  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$ . This allows to express  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$  in terms of the initial condition  $\mathcal{D} \langle \mathbf{Y}(\tau) \rangle$ . Then, substituting it into Eq. (23), one obtains a closed integro-differential equation for  $\langle \mathbf{Y} \rangle$ .

A formal solution for  $\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  is obtained from Eq. (25) in the form

$$\frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} = \mathcal{K}(t, \tau) \mathcal{D} \mathbf{Y}(\tau), \quad (28)$$

where the evolution operator

$$\mathcal{K}(t, \tau) \equiv \mathbf{T} \exp \int_\tau^t d\tau' (\mathcal{C} + V(\tau') \mathcal{D}) \quad (29)$$

is the  $T$ -ordered exponent. Disorder average of Eq. (28) introduces a correlator  $\langle \mathcal{K}(t, \tau) \mathcal{D} \mathbf{Y}(\tau) \rangle$ . Application of the decoupling, or mean field, approximation yields

$$\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle \approx \langle \mathcal{K}(t-\tau) \rangle \mathcal{D} \langle \mathbf{Y}(\tau) \rangle, \quad (30)$$

where we can write  $\langle \mathcal{K}(t, \tau) \rangle = \langle \mathcal{K}(t-\tau) \rangle$  due to the stationarity of  $V(t)$ . Terms, neglected in Eq. (30) are given formally by the generalization of the Furutsu-Novikov

formula for the correlation between two functionals of the random Gaussian process [1]:

$$\begin{aligned} \langle F\{V\} G\{V\} \rangle &= \langle F\{V\} \rangle \langle G\{V\} \rangle + \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \left\langle \frac{\delta^n F\{V\}}{\delta V(t_n) \cdots \delta V(t_1)} \right\rangle \times \\ &\times \left\langle \frac{\delta^n G\{V\}}{\delta V(s_n) \cdots \delta V(s_1)} \right\rangle \prod_{i=1}^n C_2(t_i - s_i) dt_i ds_i. \end{aligned} \quad (31)$$

In general case, when functionals  $F\{V\}$  and  $G\{V\}$  depend on the noise at simultaneous times, the decoupling approximation does not rely on small correlation radius, but depends on other parameters of the problem, e.g., small noise intensity [1]. The present case is special, because  $\mathbf{Y}(\tau)$  and  $\mathcal{K}(t, \tau)$  depend on  $V(t')$  at separate time intervals,  $0 < t' < \tau$  and  $\tau < t' < t$  respectively. Therefore, the decoupling approximation for  $\langle \mathcal{K}(t, \tau) \mathcal{D}\mathbf{Y}(\tau) \rangle$  is also justified for sufficiently small  $R_c$ . In particular, equation (30) becomes exact in the white noise limit (cf. Eq. (37) below). A small- $R_c$  perturbative expansion of the  $n = 1$  term in Eq. (31) shows that it can be neglected if both  $kR_c$  and  $\Lambda R_c$  are small compared to unity.

Substitution of the decoupling approximation (30) into Eq. (23) yields

$$\begin{aligned} \frac{\partial}{\partial t} \langle \mathbf{Y}(t) \rangle &= \mathcal{C} \langle \mathbf{Y}(t) \rangle + \\ &+ \mathcal{D} \int_0^t C_2(t - \tau) \langle \mathcal{K}(t - \tau) \rangle \mathcal{D} \langle \mathbf{Y}(\tau) \rangle d\tau, \end{aligned} \quad (32)$$

Note that inserting  $\langle \mathbf{Y}(t) \rangle = \langle \mathcal{K}(t) \rangle \mathbf{Y}_0$  into Eq. (32), one arrives at closed nonlinear equation for  $\langle \mathcal{K}(t) \rangle$ . Alternatively, assuming in Eq. (32) some explicit approximation for  $\langle \mathcal{K}(t - \tau) \rangle$ , which will be defined later, one obtains closed linear equation for  $\langle \mathbf{Y}(t) \rangle$ . It can be solved in a standard way by the Laplace transform, while its large time ( $t \gg R_c$ ) eigenvalues can be determined by substituting the asymptotic solution in the form  $\langle \mathbf{Y}(t) \rangle = \mathbf{Y}_\infty e^{4\tilde{\Lambda}t}$ , where  $\mathbf{Y}_\infty$  is a constant vector and  $\tilde{\Lambda}$  is an eigenvalue to be found. For  $t \gg R_c$ , owing to the decay of the correlation function  $C_2(t - \tau)$ , the lower limit of integration in Eq. (32) can be replaced with  $-\infty$ . Then, changing the integration variable as  $\tau \rightarrow (t - \tau)$ , one obtains a stationary eigenvalue problem, from which the generalized LE  $\Lambda$  is found as the largest real root of the characteristic equation

$$\det \left[ \mathcal{C} + \mathcal{D} \left( \int_0^\infty C_2(\tau) \langle \mathcal{K}(\tau) \rangle e^{-4\tilde{\Lambda}\tau} d\tau \right) \mathcal{D} - 4\tilde{\Lambda} \right] = 0. \quad (33)$$

Equation (33) makes sense only when  $g \equiv 2 \int_0^{+\infty} C_2(x) dx$  is finite, which is satisfied by assumption (7). This condition is important, since the correct asymptotic growth of  $\langle \mathcal{K}(\tau) \rangle$  is proportional to  $e^{4\Lambda\tau}$  (see note after Eq. (25)). Therefore, for large  $\tau$ , it should be exactly canceled by the factor  $e^{-4\tilde{\Lambda}\tau}$  in Eq. (33), when  $\tilde{\Lambda}$  is equal to  $\Lambda$ .

### C. White noise approximation for $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$

For  $\langle \mathcal{K}(\tau) \rangle$  in Eq. (33) we apply a standard white noise approximation [36] by replacing the true correlation function  $C_2(t - \tau')$  in equation (26) for  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$  with its white noise limit  $g\delta(t - \tau')$ , where  $g = 2R_c V_0^2$ , cf. Eq. (8). The integral in Eq. (26) becomes

$$\begin{aligned} \int_0^t C_2(t - \tau') \mathcal{D} \left\langle \frac{\delta^2 \mathbf{Y}(t)}{\delta V(\tau') \delta V(\tau)} \right\rangle d\tau' \approx \\ \mathcal{D}^2 \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle \int_0^t C_2(t - \tau') d\tau', \end{aligned} \quad (34)$$

where the initial condition  $\frac{\delta^2 \mathbf{Y}(t^+)}{\delta V(t) \delta V(\tau)} = \mathcal{D} \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)}$  was used (cf. Eq. (25)). Approximation (34) is justified when  $\left\langle \frac{\delta^2 \mathbf{Y}(t)}{\delta V(\tau') \delta V(\tau)} \right\rangle$  changes slowly on the scale of  $R_c$ . Since relevant characteristic scales of the problem are wave number  $k = \sqrt{|\epsilon|}$  and the generalized LE  $\Lambda$ , the conditions to be fulfilled are

$$kR_c \ll 1 \text{ and } \Lambda R_c \ll 1, \quad (35)$$

i.e. the same conditions as for the decoupling approximation (30). Inserting the white noise approximation (34) into Eq. (26) yields the following closed equation for  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$ :

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle &= \left[ \mathcal{C} + \frac{g}{2} \mathcal{D}^2 \right] \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle, \\ \left\langle \frac{\delta \mathbf{Y}(\tau^+)}{\delta V(\tau)} \right\rangle &= \mathcal{D} \langle \mathbf{Y}(\tau) \rangle, \end{aligned} \quad (36)$$

which coincides with Eq. (A1), obtained for  $\langle \mathbf{Y}(t) \rangle$  in the white noise model (Appendix A). The solution of Eq. (36) is

$$\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle = \langle \mathcal{K}_{wn}(t - \tau) \rangle \mathcal{D} \langle \mathbf{Y}(\tau) \rangle, \quad (37)$$

where  $\langle \mathcal{K}_{wn}(t - \tau) \rangle \equiv e^{\mathcal{M}(\epsilon, g)(t - \tau)}$  and matrix  $\mathcal{M}(\epsilon, g) = \mathcal{C} + \frac{g}{2} \mathcal{D}^2$  is given in Eq. (A2). Thus, decoupling (30) is obtained automatically within the white noise approximation, whereas the mean propagator in (30) is approximated by the exponent

$$\langle \mathcal{K}(t) \rangle \approx \langle \mathcal{K}_{wn}(t) \rangle \equiv e^{\mathcal{M}(\epsilon, g)t}. \quad (38)$$

For weak disorder, using approximation  $\mathcal{M}(\epsilon, g) \approx \mathcal{C}$  in the above expression for  $\langle \mathcal{K}_{wn}(t) \rangle$ , and substituting it into Eq. (33), one recovers the Born approximation (B3), which is valid for an arbitrary  $R_c$ . It follows that validity condition  $kR_c \ll 1$  is relaxed in the limit of weak disorder (while the second one,  $\Lambda R_c \ll 1$ , is usually fulfilled).

Substitution of Eq. (38) into the eigenvalue equation (33) yields an integral  $\int_0^\infty C_2(\tau) e^{\mathcal{M}(\epsilon, g)\tau} e^{-4\Lambda\tau} d\tau$ , which

is calculated by diagonalizing matrix  $\mathcal{M}(\epsilon, g)$ . The latter is not a normal matrix ( $\mathcal{M}\mathcal{M}^\dagger \neq \mathcal{M}^\dagger\mathcal{M}$ ), and can be written as [39]

$$\mathcal{Q}_L \mathcal{M}(\epsilon, g) \mathcal{Q}_R = \tilde{\mathcal{M}}(\epsilon, g) = 4 \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix}, \quad (39)$$

where eigenvalues  $\Lambda_i = \Lambda_i(\epsilon, g)$  are given in Eq. (A5), while matrices  $\mathcal{Q}_L = (u_1^L, u_2^L, u_3^L)^\dagger$  and  $\mathcal{Q}_R = (u_1^R, u_2^R, u_3^R) = \mathcal{Q}_L^{-1}$  are composed of the left and the right eigenvectors of  $\mathcal{M}(\epsilon, g)$ , given in Eq. (A6). Inserting the white noise approximation (38) into Eq. (33) and using parametrization of the correlation function, Eqs. (6) and (9), one obtains

$$\det \left[ \mathcal{C} + \frac{g}{2} \mathcal{D} \mathcal{Q}_R \mathcal{H} \mathcal{Q}_L \mathcal{D} - 4\tilde{\Lambda} \right] = 0, \quad (40)$$

where the diagonal matrix

$$\mathcal{H} = h \left( R_c \left[ 4\Lambda - \tilde{\mathcal{M}}(\epsilon, g) \right] \right), \quad (41)$$

is expressed in terms of the Laplace transform of the dimensionless correlation function

$$h(z) = \int_0^\infty \Gamma(s) e^{-zs} ds, \quad z \in \mathbb{C}. \quad (42)$$

Note that, if the dimensionless correlation function  $\Gamma(s)$  decays slower than exponentially, then  $h(z)$  is defined only for  $\text{Re } z \geq 0$ . As a consequence, the small- $R_c$  approximation will turn out to be inapplicable in the case of slowly-decaying (e.g. by power-law) correlation functions. This case should be treated with some care and will be discussed later in this section.

From now on, we restrict ourselves to the case  $\epsilon > -\frac{3}{4}g^{2/3}$ . In this regime, eigenvalues of  $\mathcal{M}(\epsilon, g)$  satisfy  $\Lambda_3 = \Lambda_2^*$  (cf. (A5)), and matrix  $\mathcal{H}$  can be written as

$$\mathcal{H} = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_r + ih_i & 0 \\ 0 & 0 & h_r - ih_i \end{bmatrix}, \quad (43)$$

where

$$\begin{aligned} h_1 &= h \left( 4R_c \left[ \tilde{\Lambda} - \Lambda_1 \right] \right), \quad h_r = \text{Re } h \left( 4R_c \left[ \tilde{\Lambda} - \Lambda_2 \right] \right), \\ h_i &= \text{Im } h \left( 4R_c \left[ \tilde{\Lambda} - \Lambda_2 \right] \right), \end{aligned} \quad (44)$$

and  $h(z)$  is defined in Eq. (42). Then, the matrix product in Eq. (40) becomes

$$\mathcal{D} \mathcal{Q}_R \mathcal{H} \mathcal{Q}_L \mathcal{D} = \frac{1}{12\Lambda_1^2 + \epsilon} \begin{bmatrix} 0 & 0 & 0 \\ f_{12} & f_{22} & 0 \\ f_{13} & f_{12} & 0 \end{bmatrix}, \quad (45)$$

Correlation	$\Gamma(x)$	$h(z)$
Exponential	$e^{-x}$	$(1+z)^{-1}$
Gaussian	$e^{-\pi x^2/4}$	$e^{z^2/\pi} \text{erfc} \frac{z}{\sqrt{\pi}}$
"Speckle"	$\text{sinc}^2 \frac{x}{2}$	$\frac{2}{\pi} \arctan \frac{\pi}{z} - \frac{z}{\pi^2} \ln \frac{z^2 + \pi^2}{z^2}, \quad \text{Re } z \geq 0$

TABLE I: Dimensionless correlation function  $\Gamma(x)$  and its Laplace transform  $h(z)$ , Eq. (42), for the models of disorder implemented in simulations.

where

$$\begin{aligned} f_{12} &= 2^{3/2} \Lambda_1 (h_1 - h_r) + \sqrt{2} \frac{6\Lambda_1^2 + \epsilon}{\sqrt{3\Lambda_1^2 + \epsilon}} h_i, \\ f_{22} &= h_1 - h_r - \frac{3\Lambda_1}{\sqrt{3\Lambda_1^2 + \epsilon}} h_i, \\ f_{13} &= 8\Lambda_1^2 (h_1 + 2h_r) + 2\epsilon h_r + \frac{2\Lambda_1 \epsilon h_i}{\sqrt{3\Lambda_1^2 + \epsilon}}. \end{aligned} \quad (46)$$

Finally, inserting matrix (45) into the eigenvalue equation (40)

$$\det \left[ \mathcal{C} + \frac{g}{2} \mathcal{D} \mathcal{Q}_R \mathcal{H} \mathcal{Q}_L \mathcal{D} - 4\tilde{\Lambda} \right] = 0, \quad (47)$$

one calculates the generalized LE  $\Lambda$  as the largest real-valued root of the determinant.

The roots of the corresponding non-linear equation can only be found numerically. Simple analytical expressions, however, can be obtained in some limiting cases. In weak disorder,  $\Lambda_1^2 \ll \epsilon$ , one recovers the Born approximation (B3), as already discussed after Eq. (38). In sufficiently strong disorder, such that  $\Lambda_1^2 \gg \epsilon$ , one can discard  $\epsilon$  in Eqs. (46), which simplifies the expressions. The resulting eigenvalue equation can be solved analytically in the limits  $\Lambda_1 R_c \ll 1$  and  $\Lambda_1 R_c \gg 1$ . The first condition,  $\Lambda_1 R_c \ll 1$ , corresponds to the white noise limit and yields  $\Lambda \equiv \Lambda_1$ , as expected. In the second limit,  $\Lambda_1 R_c \gg 1$ , which is beyond validity of the small- $R_c$  approximation and is considered formally, one obtains  $\Lambda \approx \Lambda_1$  (more specifically  $1 < \Lambda/\Lambda_1 < 1.16$ ). Between these two limits of small and large  $\Lambda_1 R_c$ , one still expects  $\Lambda \approx \Lambda_1$ , which was also found solving the eigenvalue equation numerically for the correlation functions listed in Table I. Thus, in strong disorder,  $\Lambda_1^2 \gg \epsilon$ , we obtain  $\Lambda \approx \Lambda_1$  irrespective of the form of the disorder correlation function. This result was also obtained in semiclassical approximation and in numerical simulations discussed below. According to Eqs. (A4) and (9), the above strong disorder condition can be reexpressed as  $|\epsilon|^{3/2} \ll V_0^2 R_c$ .

#### D. A regularized white noise approximation

In what follows, we refer to the approximation method described in the previous subsection, Eqs. (45)-(47), as the standard white noise approximation (SWNA). As already noted after Eq. (40), this method is not applicable to slowly decaying correlations. We now discuss this

point in some more detail and propose a simple regularization to extend the applicability of the approximation.

The white noise propagator  $\langle \mathcal{K}_{wn}(t) \rangle$ , involved in approximation (37) for  $\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \rangle$ , has the largest eigenvalue  $e^{4\Lambda_1(\epsilon, g)t}$ , where  $\Lambda_1(\epsilon, g)$  is the generalized LE for the uncorrelated disorder (see Appendix A). As follows from the Born approximation (B3), it is usually larger than the generalized LE  $\Lambda$  for the correlated potential with the same intensity  $g$ , at least for sufficiently weak disorder. Therefore, if correlation function  $C_2(\tau)$  decays slower than  $e^{-4(\Lambda_1(\epsilon, g) - \Lambda)\tau}$ , then the integral in the eigenvalue equation (33), which can be estimated as

$$\int_0^\infty C_2(\tau) \langle \mathcal{K}_{wn}(t) \rangle e^{-4\tilde{\Lambda}\tau} d\tau \sim \int_0^\infty C_2(\tau) e^{4(\Lambda_1(\epsilon, g) - \tilde{\Lambda})\tau} d\tau, \quad (48)$$

diverges for  $\tilde{\Lambda} = \Lambda$ . In the final expressions, Eqs. (40)-(46), this potential divergence resides in function  $h_1 = \int_0^\infty \Gamma(\tau) e^{4(\Lambda_1(\epsilon, g) - \tilde{\Lambda})\tau} d\tau$ , defined by Eqs. (42) and (44). Thus, for slowly, e.g. sub-exponentially, decaying correlations,  $\Lambda$  can not be a solution of the eigenvalue equation and the standard white noise approximation is a priori inapplicable. Divergence of the integral in Eq. (48) as an artifact of the white noise approximation (37), which yields a wrong large-time asymptotics  $\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \rangle \sim e^{4\Lambda_1(\epsilon, g)(t-\tau)}$ . Indeed, as noted after Eq. (25), the true asymptotics of the moment of the functional derivative is  $\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \rangle \sim e^{4\Lambda(t-\tau)}$ , i.e. the same as for  $\langle \mathbf{Y}(t) \rangle$ . Therefore, if  $\langle \mathcal{K}_{wn}(t) \rangle$  in (48) had a "correct" asymptotics, then the integral would converge at  $\tilde{\Lambda} = \Lambda$  as long as  $\int_0^\infty C_2(\tau) d\tau \equiv \frac{g}{2} < \infty$ . Thus, some kind of a regularization is required for  $\tau \gg R_c$ . One possibility is to introduce an upper cutoff of the order of  $R_c$  into the integral for  $h_1$ . Such cutoff was effectively used in Ref. [23] for another scheme of a closure of Eq. (23). A more simple regularization is just to set  $h_1 = \int_0^\infty \Gamma(\tau) d\tau = 1$ , which amounts to a self-consistent modification of the white noise propagator  $\langle \mathcal{K}_{wn}(t) \rangle$  in the approximation (38) by replacing its largest eigenvalue  $e^{4\Lambda_1(\epsilon, g)t}$  with  $e^{4\Lambda t}$ . We use the second approach and refer to it as a "regularized white noise approximation" (RWNA). Thus, the regularized white noise approximation is given by Eqs. (45)-(47), with  $h_1$  replaced by unity. Note, that this method can be regarded as applying the decoupling approximation (30) with the corresponding effective propagator, different from that in Eq. (38).

### E. Numerical test of small- $R_c$ approximation

The standard and the regularized white noise approximations are compared to the direct numerical simulation for the three types of disorder correlations given in Table I. Analytical values of  $\Lambda$  are obtained by finding numerically the largest real-valued root of the determinant

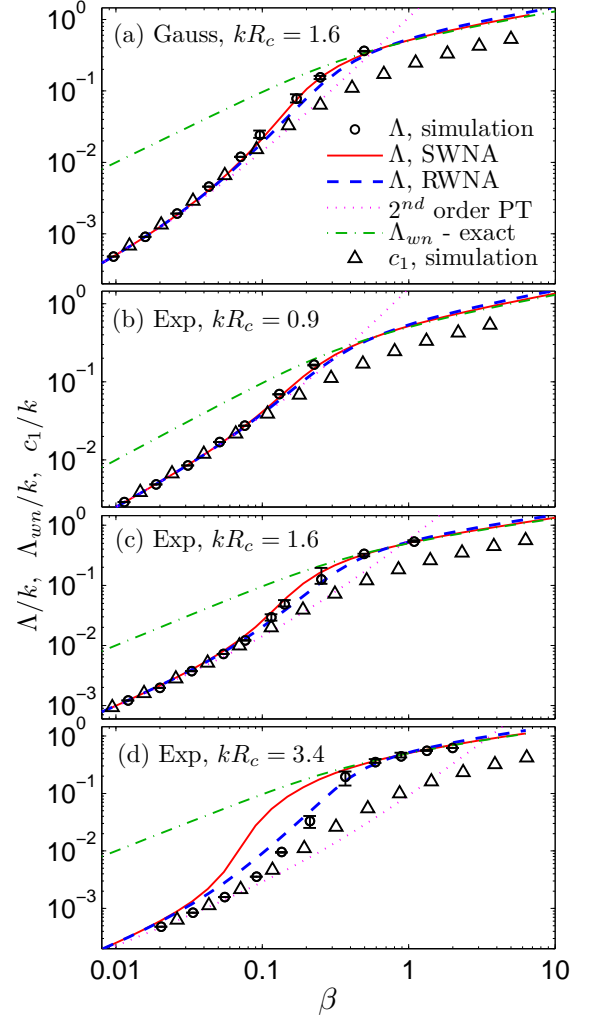


FIG. 1: (color online) The standard (SWNA) and the regularized (RWNA) white noise approximations for the generalized LE  $\Lambda$  are compared to numerical simulation for (a) Gaussian and (b-d) exponential correlation functions (Table I). Legends in (a) apply to all panels. GLE, normalized to the wavevector  $k$ , is plotted against the dimensionless disorder intensity  $\beta$ , Eq. (49). Error bars indicate uncertainty of the Monte-Carlo calculation of  $\Lambda$ . For comparison, we also plot  $2^{nd}$  order (i.e., one order beyond the Born approx.) perturbative calculation of  $\Lambda$  due to Ref. [22], and the exact GLE  $\Lambda_{wn}$  for  $\delta$ -correlated disorder given by  $\Lambda_1$  in Eq. (A5). In addition, LE  $c_1$ , obtained by numerical simulation, is plotted with triangles. As expected for Gaussian disorder [20],  $\Lambda$  and  $c_1$  coincide in weak disorder as long as  $2^{nd}$  order perturbation theory is valid. In strong disorder,  $\beta \sim 1$ , GLE for correlated disorder coincides with  $\Lambda_{wn}(\beta)$ .

(40). The Monte-Carlo simulations were done for the tight-binding model (C1) with energy  $E = -2 \cos k$  near the band edge, which translates to the continuous model (5) with a positive energy  $\epsilon = k^2$ . Further details of the numerical simulations are given in Appendix C.

First, we examine models of disorder with the quickly



decaying exponential and Gaussian correlation functions (Table I). The results are compared to the Monte-Carlo simulation in Fig. 1. The error bars indicate uncertainty of the Monte-Carlo calculation of the generalized LE, which is explained in Appendix C. the generalized LE  $\Lambda$  is plotted for different values of  $kR_c$  as a function of the dimensionless noise intensity

$$\beta \equiv \frac{gk}{8\epsilon^2} = \frac{V_0^2 k R_c}{4\epsilon^2}, \quad (49)$$

where  $g$  is defined in Eq. (9). Note that  $\beta = \Lambda/k$  in the Born approximation (B3) for the white noise disorder.

The standard and the regularized white noise approximations yield close results for  $kR_c \lesssim 1$ , while Difference between them increases with  $R_c$  and, e.g., for the exponential correlation with  $kR_c = 3.4$ , it becomes of the order of ten (Fig. 1(d)). Comparison with the simulation results shows that both approximations are good for  $kR_c \lesssim 1$  (Fig. 1(a-c)), but deteriorate with further increase of  $kR_c$ , as expected. For example, for the exponential correlation with  $kR_c = 3.4$ , the result of the standard approximation and the simulated values of  $\Lambda$  differ already by a factor of ten at moderately strong disorder,  $\beta \sim 0.1$  (Fig. 1(d)). The regularized white noise approximation seems to be somewhat better, though the improvement is inconclusive.

The main advantage of the regularized white noise approximation is its applicability to the long-range correlations, as demonstrated in Fig. 2. As an example, we use "speckle" correlation function  $\Gamma(x) = \text{sinc}^2 \frac{x}{2}$ , where  $\text{sinc } x \equiv \frac{\sin \pi x}{\pi x}$ , which describes correlation of the intensity in some quasi-one-dimensional laser speckle patterns [40]. The latter are used to create disordered potentials in experiments with cold atoms [17]. Note, however, that our Gaussian model does not describe a true speckle intensity, which exhibits highly non-Gaussian fluctuations. This correlation function has a remarkable property that the corresponding power spectrum, given by the Fourier transform of the correlation function  $2V_0^2 \int_0^\infty dx \Gamma(x/R_c) \cos qx$ , is proportional to the "tent" function  $(1 - |qR_c/\pi|) \Theta(1 - |qR_c/\pi|)$  and vanishes for  $|qR_c| > \pi$ . As a results, the Born approximation for both LE  $c_1$  and the generalized LE  $\Lambda$ , Eq. (B3), vanishes for  $2kR_c > \pi$ . In view of this fact, the regularized white noise approximation is tested below and above this threshold for  $2kR_c$  equal to  $0.9\pi$  and  $1.03\pi$  respectively. In both cases we find agreement with the simulation results within the factor of the order of unity (Fig. 2).

Let us summarize the above comparison of the analytical and the numerical results. The suggested simple regularization allows extension of the standard white noise closure approximation to the cases of slowly decaying correlations. In the regime of moderate disorder,  $0.1 \lesssim \beta \lesssim 1$ , white noise approximation is applicable for  $kR_c \lesssim 1$  (we consider  $\epsilon = k^2 > 0$ ), in accord with the formal requirement  $kR_c \ll 1$  (cf. Eq. (35)). In the limit of weak disorder,  $\beta \ll 1$ , our approximation coincides with the perturbation theory (B3), i.e. becomes

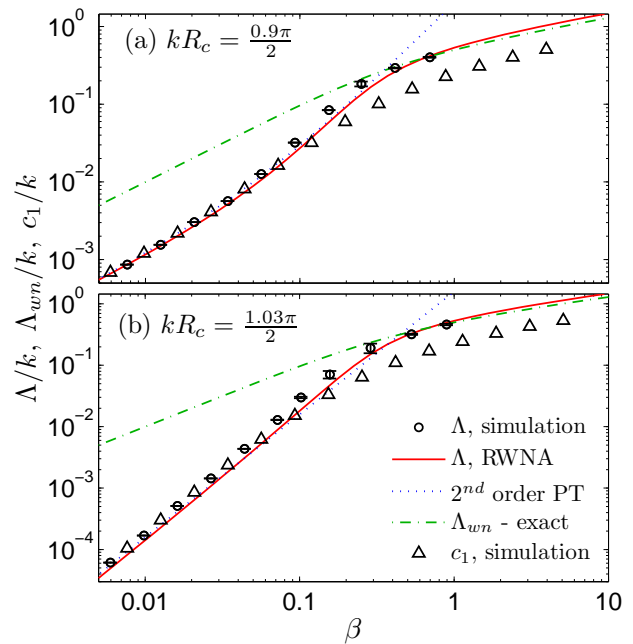


FIG. 2: (color online) The regularized white noise approximation (RWNA) and the numerical simulation of  $\Lambda$  are compared for the power law "speckle" correlation (See Table I) with (a)  $kR_c = 0.9\pi/2$  and (b)  $kR_c = 1.03\pi/2$  (legends in (b) apply to both panels). Error bars indicate uncertainty of the numerical calculation of  $\Lambda$ . Like in Fig. 1,  $\Lambda_{wn}(\beta)$  and the  $2^{nd}$  order perturbative calculation of  $\Lambda$  due to Ref. [22] are given for reference. Triangles represent numerical result for LE  $c_1$ . As in Fig. 1,  $\Lambda$  coincides with  $c_1$  in weak disorder, and with  $\Lambda_{wn}(\beta)$  in strong disorder.

exact up to weak disorder corrections. Thus, for  $\beta \ll 1$ , the validity condition  $kR_c \ll 1$  is relaxed (cf. discussion after Eq. (38)). Finally, for strong disorder,  $\beta > 1$ , numerical data coincide with and, thus, confirm the analytical result  $\Lambda(\beta) \approx \Lambda_1(\beta) \equiv \Lambda_{wn}(\beta)$ , obtained for  $|\epsilon|^{3/2} \ll V_0^2 R_c$ , i.e.  $\beta \gg 1$ , at the end of Sec. III C. This is inspite of the fact that for  $\beta > 1$  in our simulations we have  $\Lambda R_c \gtrsim 1$ , which violates the white noise approximation validity condition  $\Lambda R_c \ll 1$ , Eq. (35). In the next section, however, the same relation  $\Lambda(\beta) \approx \Lambda_1(\beta)$  is obtained in the regime  $\Lambda R_c \gg 1$  *specifically* for Gaussian disorder.

#### IV. SEMICLASSICAL REGIME OF STRONG DISORDER

In this section we analytically calculate cumulant coefficients  $c_n$ , defined by Eq. (12), in the regime when conditions  $\epsilon < V_0$  and  $R_c \sqrt{V_0 - \epsilon} \gg 1$  are fulfilled, where  $V_0^2$  is the noise variance introduced in Eq. (6). An opposite regime,  $R_c \sqrt{V_0 - \epsilon} \ll 1$ , corresponds to the white noise limit, for which the cumulants were calculated in Ref. [5]. We assume that  $V_0$  and  $R_c$  are typical height

and width of the random barriers, therefore probability of a tunneling through a single typical barrier is exponentially small under the above conditions. In this case, as argued in Ref. [25], transport is dominated by the under-barrier tunneling and interference effects are negligible. Since  $R_c$  is a spatial scale of the disorder  $V(x)$ , the second condition,  $R_c\sqrt{V_0 - \epsilon} \gg 1$ , justifies the semiclassical approximation for the under-barrier tunneling. Thus, an exponential growth rate of the solution amplitude  $A(x) = \sqrt{\psi^2 + \psi'^2}$  can be estimated as a magnitude of the imaginary part of the semiclassical action:

$$\ln |A(L)| = \int_0^L \kappa(x) dx. \quad (50)$$

Here  $\kappa(x) = \Theta_\epsilon(V) \sqrt{V(x) - \epsilon}$  is an imaginary part of the complex momentum, where  $\Theta_\epsilon(V) \equiv \Theta(V - \epsilon)$  is a short notation for the unit step function. Function  $\Theta_\epsilon(V)$  selects regions where the energy is below the disorder barriers and solution  $\psi(x)$  grows exponentially, while the remaining regions, where  $\psi(x)$  has an oscillatory behavior, are discarded.

Semiclassical approximation (50) directly relates distribution of  $\ln |A(L)|$  to the statistical properties of the above-threshold excursions of the random process  $V(x)$ . Disorder average of Eq. (50) yields LE or the first cumulant coefficient

$$c_1 = \langle \kappa \rangle = \langle \kappa \rangle_\epsilon \langle \Theta_\epsilon(V) \rangle, \quad (51)$$

where  $\langle \kappa \rangle_\epsilon \equiv \int_\epsilon^\infty dV \sqrt{V - \epsilon} P_\epsilon(V)$  is a conditional average for  $V > \epsilon$  over the distribution  $P_\epsilon(V) \equiv \Theta_\epsilon(V) P(V) / \int_\epsilon^\infty dV P(V)$ . Here  $P(V)$  is distribution of the disorder values at a single position.

Using semiclassical approximation (50), higher cumulant coefficients  $c_n$  can be expressed in terms of the corresponding joint cumulants of  $\kappa(x)$ :

$$c_n = \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L \cdots \int_0^L \prod_{i=1}^n dx_i \left\langle \left\langle \prod_{i=1}^n \kappa(x_i) \right\rangle \right\rangle. \quad (52)$$

Since  $V(x)$  is a stationary process, the joint cumulants  $\langle\langle \cdots \rangle\rangle$  in Eq. (52) depend only on the coordinate difference. The value of the cumulants is maximal when all the coordinates coincide, and decays on the scale  $R_c$  with a distance between the points. This is because  $R_c$  is the only spatial scale in our model (6). Thus, shifting all the coordinates in the cumulant by  $x_n$  and assuming  $L \gg R_c$ , the integral in (51) can be approximated as

$$c_n \approx \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n-1} dx_i \left\langle \left\langle \kappa(0) \prod_{i=1}^{n-1} \kappa(x_i) \right\rangle \right\rangle, \quad (53)$$

where we have neglected the boundary effects at the corners  $x_1 = x_2 = \dots = x_n = 0, L$  and extended the integration to  $\pm\infty$ . For the Gaussian process  $V(x)$ , an analytical calculation of the joint cumulants of  $\kappa(x)$  is

quite involved. It is, however, possible to obtain a simple estimate of the multiple integral in (53). As an example, consider the two-point correlator of the regular step function  $\Theta(V(x))$ , which can be calculated analytically [41], and is given by

$$\frac{\langle\langle \Theta(V(x)) \Theta(V(x')) \rangle\rangle}{\langle\langle \Theta^2(V) \rangle\rangle} = \frac{2}{\pi} \arcsin \left[ \Gamma \left( \frac{x - x'}{R_c} \right) \right], \quad (54)$$

where the dimensionless correlation function  $\Gamma(x)$  was defined in Eq. (6). As expected, the two-point correlator of  $\Theta(V(x))$  decays in the same manner as  $\Gamma(x)$ . Due to the normalization conditions (7), the integral of the expression in Eq. (54) is of the order of  $R_c$ . Similarly, any  $n$ -order cumulant in (53) has maximum value  $\langle\langle \kappa^n(0) \rangle\rangle$  and decays with a distance from the origin on the scale  $R_c$ . Therefore, rescaling the integration variables in (53) by  $R_c$ , the remaining dimensionless integral can be grossly estimated as a volume of the  $(n-1)$ -dimensional unit sphere  $S_{n-1}$ . This gives the following estimate

$$c_n = B_n \left( \frac{\epsilon}{V_0} \right) S_{n-1} R_c^{n-1} \langle\langle \kappa^n \rangle\rangle, \quad (55)$$

where functions  $B_n(\epsilon/V_0)$  are of the order of unity (according to Eq. (51),  $B_1 \equiv 1$ ), and  $\langle\langle \kappa^n \rangle\rangle \equiv \langle\langle \kappa^n(x) \rangle\rangle$ , which is independent of the position  $x$ . Coefficients  $B_{n>1}(\epsilon/V_0)$  compensate for the approximation of the  $(n-1)$ -dimensional integral by the volume of sphere and depend weakly on the specific form of the correlation function of  $V(x)$  and on the ratio  $\epsilon/V_0$ . Numerical values of  $B_n(\epsilon/V_0)$  can be found in computer simulation by calculating statistics of the quantity on the right hand side of Eq. (50), and fitting it to Eq. (55). As an example, we obtain  $B_2(0) \approx 0.9$ ,  $B_2(\frac{1}{3}) \approx 0.8$  and  $B_3(0) \approx 1.6$ ,  $B_3(\frac{1}{3}) \approx 1.2$  for all three types of correlations given in Table I.

Combining Eqs. (51) and (55), one obtains the following relations

$$\frac{c_n}{c_1} = S_{n-1} B_n \left( \frac{\epsilon}{V_0} \right) f_n \left( \frac{\epsilon}{V_0} \right) (c_1 R_c)^{n-1}, \quad (56)$$

where  $f_n \left( \frac{\epsilon}{V_0} \right) \equiv \frac{\langle\langle \kappa^n \rangle\rangle}{\langle\langle \kappa \rangle\rangle^n}$  are dimensionless functions, whose specific form depends only on the disorder distribution  $P(V)$ . This result should be contrasted with the weak disorder relations (4).

Using the fact that  $\Theta_\epsilon(V)$  is either 0 or 1, it is convenient to express cumulants of  $\kappa(x)$  in terms of  $\langle\langle \Theta_\epsilon^n(V) \rangle\rangle$  and  $\langle\langle \kappa^n \rangle\rangle_\epsilon$ , where  $\langle\langle \cdots \rangle\rangle_\epsilon$  denotes a "conditional cumulant", calculated with distribution  $P_\epsilon(V)$  [see Appendix

D]. For example, for  $n = 2, 3$  one obtains

$$\begin{aligned} f_2\left(\frac{\epsilon}{V_0}\right) &= \frac{\langle\langle\Theta_\epsilon^2(V)\rangle\rangle}{\langle\Theta_\epsilon(V)\rangle^2} + \frac{1}{\langle\Theta_\epsilon(V)\rangle} \frac{\langle\langle\kappa^2\rangle\rangle_\epsilon}{\langle\kappa\rangle_\epsilon^2}, \\ f_3\left(\frac{\epsilon}{V_0}\right) &= \frac{1}{\langle\Theta_\epsilon(V)\rangle^2} \frac{\langle\langle\kappa^3\rangle\rangle_\epsilon}{\langle\kappa\rangle_\epsilon^3} + \\ &+ 3 \frac{\langle\langle\Theta_\epsilon^2(V)\rangle\rangle}{\langle\Theta_\epsilon(V)\rangle^3} \frac{\langle\langle\kappa^2\rangle\rangle_\epsilon}{\langle\kappa\rangle_\epsilon^2} + \frac{\langle\langle\Theta_\epsilon^3(V)\rangle\rangle}{\langle\Theta_\epsilon(V)\rangle^3} \end{aligned} \quad (57)$$

This expansion is helpful, since it shows separate contributions of the fluctuation of the indicator function  $\Theta_\epsilon(V)$  and of the barrier height fluctuation above the level  $\epsilon$ .

For the Gaussian distribution,  $P(V) = (2\pi V_0^2)^{-1/2} e^{-V^2/2V_0^2}$ , explicit expression for  $c_1$ ,  $f_2\left(\frac{\epsilon}{V_0}\right)$  and  $f_3\left(\frac{\epsilon}{V_0}\right)$  are given in Eqs. (D3) and (D4).

Finally, the semiclassical approximation (50) can be used to calculate the generalized LE  $\Lambda$  in strong disorder limit

$$\Lambda_{sc} = \lim_{L \rightarrow \infty} \frac{1}{4L} \ln \left\langle \exp \left[ 2 \int_0^L \kappa(x) dx \right] \right\rangle, \quad (58)$$

where the disorder average of the exponential involves functional integration over  $V(x)$ . As an example, we calculate  $\Lambda_{sc}$  for Gaussian disorder in the limit  $|\epsilon| \ll V_0$  and  $\sqrt{V_0}R_c \gg 1$ , which justifies the stationary point approximation. The latter yields

$$\Lambda_{sc} = \frac{3}{8} (2R_c V_0^2)^{1/3}. \quad (59)$$

According to Eqs. (59) and (D3), under the conditions  $|\epsilon| \ll V_0$  and  $\sqrt{V_0}R_c \gg 1$ , LE ratio  $\rho = \Lambda/c_1$  scales like  $(R_c V_0^2)^{1/3} / V_0^{1/2} \sim (c_1 R_c)^{1/3}$ . For comparison, the Gaussian part of  $\rho$ , i.e.  $\rho_G = \frac{1}{2} \left( 1 + \frac{c_2}{c_1} \right)$  [cf. Eq.(16)], scales in this regime as  $c_2/c_1 \sim c_1 R_c$ . The obtained scaling is closely related to the statistics of the disorder fluctuations and, thus, is specific to the considered Gaussian models (e.g., for strong binary disorder one expects  $\rho \sim \text{const}$ ).

Let us note that  $\Lambda_{sc}$  in Eq. (59) matches very closely the strong disorder limit of the small- $R_c$  approximation for  $\Lambda$  (Sec. III C, last paragraph). The latter reads  $1 < \Lambda/\Lambda_1(g) < 1.16$ , where  $g = 2V_0^2 R_c$  and  $\Lambda_1(g) = \frac{1}{2} \left( \frac{g}{4} \right)^{1/3}$  is the generalized LE for  $\delta$ -correlated disorder in strong disorder limit (cf. Eq.(A4)). The semiclassical result (59) gives  $\Lambda_{sc}/\Lambda_1(g) = 1.19$ , which is remarkably close to the small- $R_c$  approximation. Note that the considered limit,  $\sqrt{V_0}R_c \gg 1$ , implies that  $\Lambda R_c \gg 1$ , which is formally beyond validity of the small- $R_c$  approximation.

## V. TRANSMISSION STATISTICS

In this section we discuss properties of the transmission coefficient distribution in terms of the dimensionless

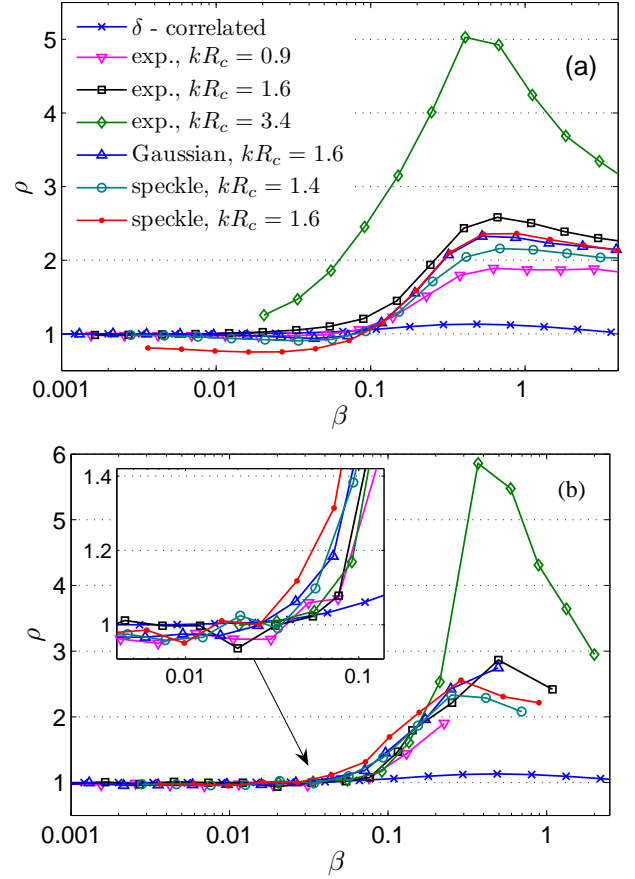


FIG. 3: (color online) LE ratio  $\rho = \Lambda/c_1$  as a function of the dimensionless disorder intensity  $\beta$  [Eq. (49)]. LE  $c_1$  is obtained by numerical simulations. GLE  $\Lambda$  is calculated using both the analytical RWNA method, panel (a), as well as numerical simulations, panel (b). Legends in (a) apply to both panels and specify type of correlation and value of  $kR_c$ . The analytical and the numerical results complement each other in different regimes of disorder. The numerics is noisy or unavailable at sufficiently large disorder intensities,  $\beta > 0.1$ , where analytical results demonstrate how  $\rho$  increases with  $kR_c$ . On the other hand, numerical results are more accurate for smaller disorder intensities,  $\beta \lesssim 0.1$ , where deviation from the limiting weak disorder value  $\rho = 1$  begins to develop, as shown in the inset of panel (b).

ratios  $c_2/c_1$ ,  $c_3/c_1$  and  $\rho = \Lambda/c_1$ , Eqs. (13) and (16). The cumulant coefficients are simulated numerically for three types of correlations listed in Table I and for the white noise model as well. For the generalized LE  $\Lambda$  we use both analytical and numerical results. Details of the numerical simulations are given in Appendix C. All calculations are performed at the same fixed value of energy,  $\epsilon = k^2 > 0$ , and for different values of the correlation radius  $R_c$  and the disorder variance  $V_0^2$ . Disorder strength is conveniently characterized by the dimensionless disorder intensity  $\beta = V_0^2 k R_c / 4\epsilon^2$ , introduced in Eq. (49). For each considered value of  $R_c$ , quantities  $\Lambda$  and  $c_n$  are studied as a function of  $V_0$  in the range from weak ( $\beta \ll 1$ ) to strong ( $\beta \sim 1$ ) disorder.

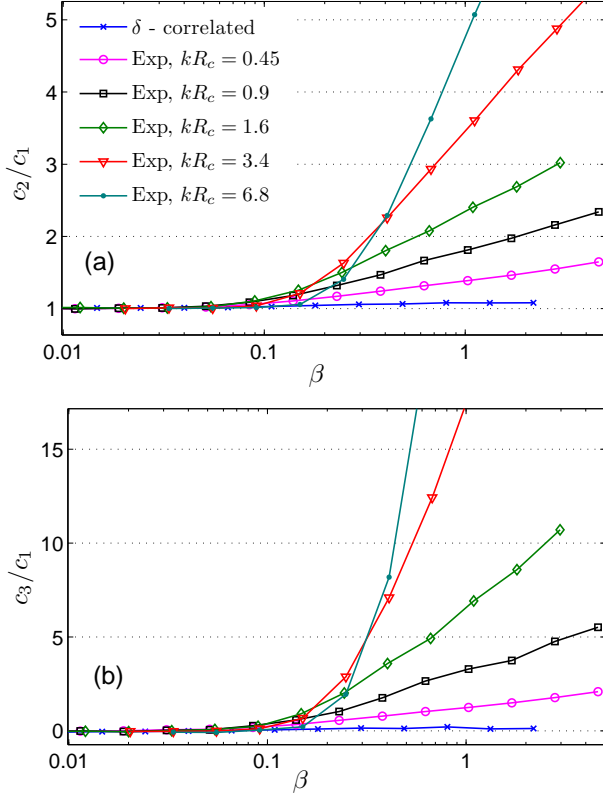


FIG. 4: (color online) Cumulant ratios  $c_2/c_1$  (upper panel) and  $c_3/c_1$  (lower panel) for the  $\delta$ -correlated and the exponentially correlated disorder plotted as a function of  $\beta$  (other models of correlation given in Table I exhibit similar behavior, cf. Fig. 6). Legends in (a) apply to both panels and specify the types of correlation and the values of  $kR_c$ .

In Figs. 3 and 4 we plot  $\rho$  and cumulant ratios  $c_2/c_1$  and  $c_3/c_1$  respectively as a function of  $\beta$ . Different data series correspond to different models of correlation and different fixed values of  $kR_c$ , as indicated in the legends. All three quantities,  $\rho$ ,  $c_2/c_1$  and  $c_3/c_1$ , exhibit qualitatively similar behavior in weak and moderate disorder ( $\beta < 1$ ). As expected from the results on the  $\delta$ -correlated disorder [3],[5], universal relations  $\rho = 1$ ,  $c_2/c_1 = 1$  and  $c_3/c_1 = 0$  (cf. Eqs.(4) and (16)) are violated beyond weak disorder. Our calculations show that deviation from these weak disorder values is strongly enhanced in the presence of correlations and increases with the correlation radius  $R_c$ . Namely, while this deviation is small compared to unity for  $\delta$ -correlated disorder, it becomes of the order of unity, or even larger, in the presence of correlations. Below we discuss the obtained results and the corresponding parametric dependences.

#### A. Scaling of the cumulant ratios $c_n/c_1$

In sufficiently strong disorder, such that  $c_1 R_c \gg 1$ , cumulant ratios  $c_n/c_1$  are described by the semiclassi-

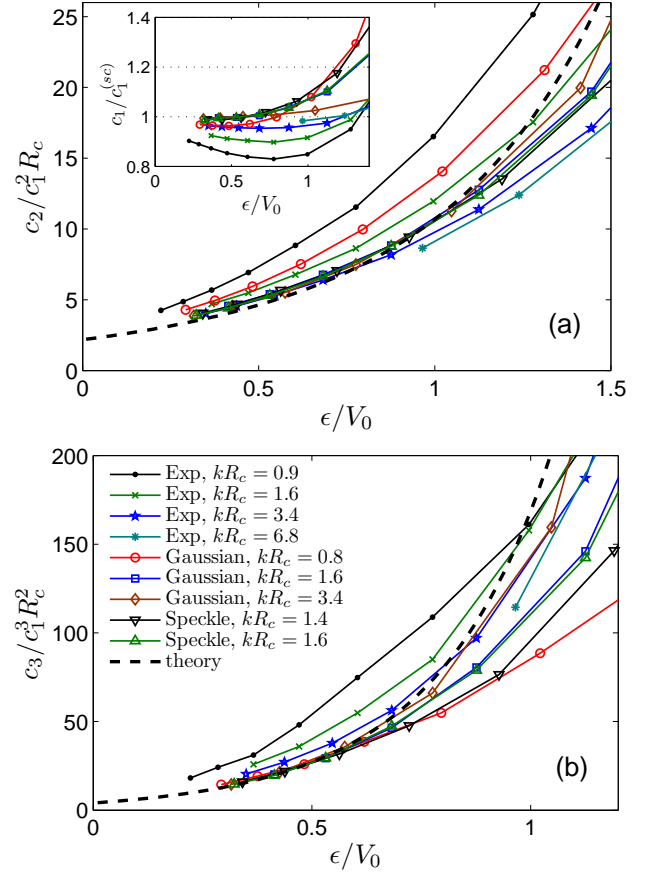


FIG. 5: (color online) Semiclassical scaling relations (57) are verified for the cumulant coefficients  $c_2$  (upper panel) and  $c_3$  (lower panel) by plotting simulated values of  $c_n/c_1^n R_c^{n-1}$  as a function of the parameter  $\epsilon/V_0$ . Legends in (b) apply to both panels. The dashed lines represent analytical result due to Eq. (57) with  $B_2 = 0.8$  and  $B_3 = 1.2$  (see text after Eq. (57)). To check the semiclassical approximation for LE  $c_1$ , Eq. (51), inset in panel (a) shows that ratio of the simulated value of  $c_1$  to its semiclassical approximation, denoted as  $c_1^{sc}$ , approaches unity for  $\epsilon/V_0 < 1$ . As expected, semiclassical approximation improves for larger values of  $R_c$  and smaller values of  $\epsilon/V_0$ .

cal relations (56). To verify these relations, in Fig. 5 we plot  $(c_n/c_1)/(c_1 R_c)^{n-1}$  for  $n = 2, 3$  as a function of  $\epsilon/V_0$  and compare it to the semiclassical prediction  $S_{n-1} B_n \left(\frac{\epsilon}{V_0}\right) f_n \left(\frac{\epsilon}{V_0}\right)$ , according to Eq. (56). For Gaussian disorder, functions  $f_n \left(\frac{\epsilon}{V_0}\right)$  are given by Eq. (D4), and we use values  $B_2 = 0.8$  and  $B_3 = 1.2$ , as explained in text after Eq. (57). Fig. 5 shows that the obtained numerical values converge to the theoretical curve for  $\epsilon/V_0 < 1$ . As expected, the semiclassical approximation improves for smaller  $\epsilon/V_0$  and larger  $R_c$ . Let us note, for clarity, that in our presentation we increase the disorder amplitude  $V_0$  keeping fixed  $\epsilon > 0$  and fixed  $R_c$ . In this parametrization, cumulant ratios  $c_n/c_1$  grow monotonically with  $\beta$ , roughly as  $(c_1 R_c)^{n-1}$  (cf. Fig. 4). If,

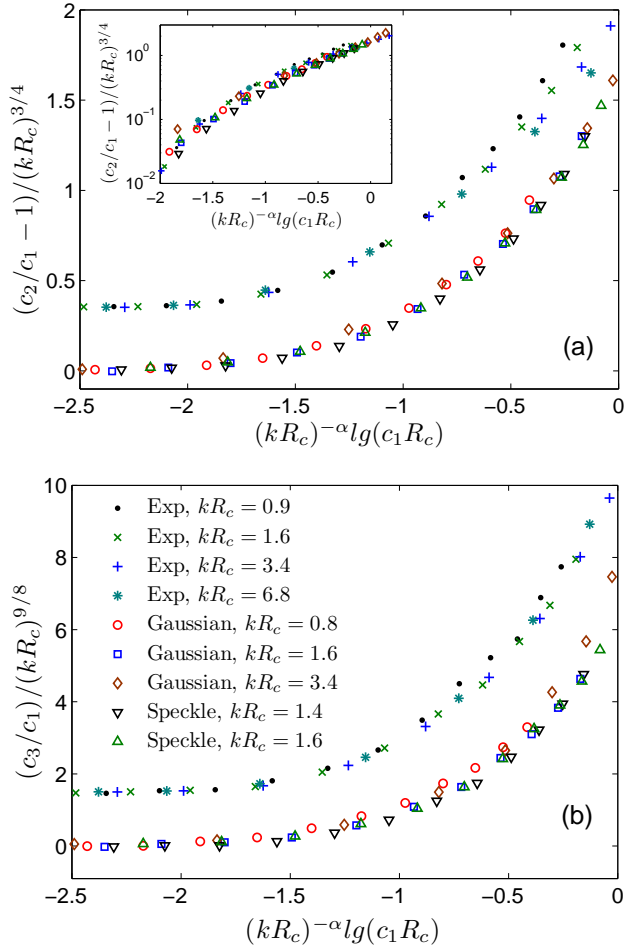


FIG. 6: (color online) Demonstration of the empirical scaling (60) for the cumulant coefficients  $c_2$  (upper panel) and  $c_3$  (lower panel). Legends in (b) apply to both panels and specify type of correlation and value of  $kR_c$ . We use  $\alpha = 0$  for the exponential correlation, and  $\alpha = 0.4$  - for the Gaussian and the speckle correlations. For better visibility, data for the exponential correlation are offset from zero by a constant (in both panels). Inset: same as in the main panel (a), but plotted in a logarithmic scale for  $y$ -axis and without the offset.

however, relative disorder strength is increased by decreasing  $\epsilon$  at fixed  $V_0$  and  $R_c$ , then ratios  $c_n/c_1$  would eventually vanish for sufficiently large and negative  $\epsilon$ , as can be seen from Eq. (56) [because  $V_0/|\epsilon|$  becomes small in this limit].

Semiclassical scaling (56) is not applicable in weak and moderate disorder. In this regime, and for  $kR_c \gtrsim 1$ , we find empirically that deviation from the weak disorder relations (4) for  $n = 2, 3$  is described by the following approximate scaling

$$\frac{c_n}{c_1} - \delta_{n,2} = (kR_c)^{\nu_n} \mu_n \left( \frac{\ln c_1 R_c}{[kR_c]^\alpha} \right), \quad (60)$$

where  $\delta_{n,2}$  is the Kronecker delta. Here  $\mu_n(x)$  are some dimensionless monotonically increasing functions, whose form depend on the type of correlation. Exponent  $\alpha$

is not universal as well, and we find that  $\alpha \approx 0$ , for the exponential, and  $\alpha \approx 0.4$  for the Gaussian and the speckle correlation functions (Table I). On the contrary, in the considered Gaussian models, we obtain  $\nu_2 \approx 3/4$  and  $\nu_3 \approx 9/8$ , irrespectively of the disorder correlation. In Fig. 6 we plot  $(c_2/c_1 - 1)/(kR_c)^{3/4}$  and  $(c_3/c_1)/(kR_c)^{9/8}$  against  $(kR_c)^{-\alpha} \lg c_1 R_c$ , which demonstrates that scaling (60) holds for a rather broad range of the considered values of  $kR_c$  (see legends) as long as  $c_1 R_c < 1$ . A logarithmic scale plot of  $(c_2/c_1 - 1)/(kR_c)^{3/4}$ , given in the inset of the upper panel, shows that this scaling holds also at small values of  $c_1 R_c$  (the same is true also for  $c_3/c_1$ ).

Let us stress that relations (60) become meaningless in the white noise limit  $kR_c \rightarrow 0$ , and are applicable only for  $kR_c \gtrsim 1$ , when deviation from the weak disorder relations (4) is dominated by the effects of the correlations. According to Eq. (60), the latter is controlled by the ratio of the correlation to the localization lengths,  $c_1 R_c$ .

As seen in Fig. (6), functions  $\mu_n$  are rather similar, though not identical, for different models of correlation. The major distinction in a specific form of relation (60) for different (Gaussian) models appears in the value of the exponent  $\alpha$ . It is interesting to note that both  $\mu_n$  and  $\alpha$  practically coincide for the models with Gaussian and speckle correlation functions. This can be related to the qualitative similarity of their power spectra, given by the Gaussian and the "tent" function respectively, which have either effective or exact cutoff of the order of  $R_c^{-1}$  (the "tent" function,  $(1 - |x|) \Theta(1 - |x|)$ , was introduced in Sec. III E). On the contrary, power spectrum of the exponentially correlated disorder is given by the slowly decaying Lorentzian function. Note that disorder power spectrum appears, e.g., in the Born approximation (B3) for  $\Lambda$  (and LE  $c_1$ ), which shows close relation between the properties of localization and those of the disorder power spectrum.

## B. Extreme fluctuation of $T^{-1}$

Beyond weak disorder, LE ratio  $\rho$  increases with the correlation parameter  $kR_c$  and, as a function of the disorder intensity, exhibits a peak [42] at moderate disorder (Fig. 3). Thus, as expressed in terms of the ratio between the mean and the typical values of the inverse transmission coefficient  $T^{-1}$ , Eq. (17), also the extreme relative fluctuation of  $T^{-1}$  is peaked at moderate disorder and is enhanced by the disorder correlations. This conclusion is consistent with our observation that statistical convergence of the Monte-Carlo simulation of the generalized LE  $\Lambda$  is most slow at moderate disorder and for larger values of  $R_c$ , as indicated by the error bars in Figs. 1 and 2. Note that relative fluctuation (17) depends exponentially on  $\rho l$ , where  $l \equiv c_1 L$ . Therefore, for  $kR_c \gtrsim 1$  and  $l > 1$ , correlations lead to the exponentially large enhancement of the extreme fluctuation of  $T^{-1}$  as com-



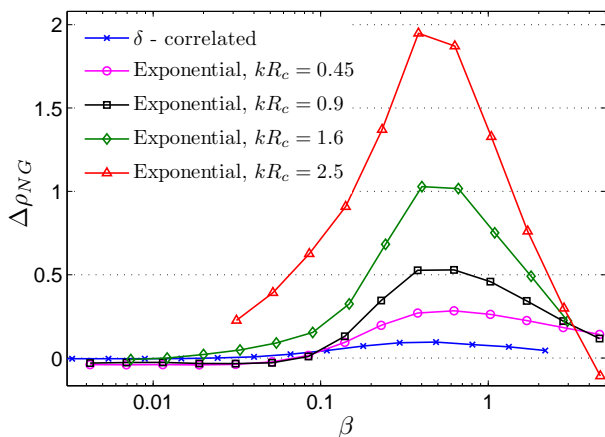


FIG. 7: (color online) The non-Gaussian part of  $\rho$ ,  $\Delta\rho_{NG} = \rho - \rho_G$ , where  $\rho_G = (1 + c_2/c_1)/2$ , as a function of  $\beta$ . The shown data correspond to the  $\delta$ -correlated and exponentially correlated disorder with different values of  $kR_c$ , as indicated in the legends.  $\Lambda$  is calculated analytically using the RWNA method. As noted in Sec. III E, approximation error of this method becomes significant for  $kR_c > 1$ . In the presented figure, this does results in false values for  $\beta \lesssim 0.1$  (simulated values are much closer to zero). However, the approximation error is not critical in the region of main interest,  $0.1 \lesssim \beta \lesssim 1$ , where it tends to underestimate, rather than overestimate, the actual values of  $\Delta\rho_{NG}$ . Thus, the presented effect is not an artifact of the approximation.

pared to the case of the  $\delta$ -correlated disorder.

It is instructive to compare  $\rho$  to its Gaussian part  $\rho_G = \frac{1}{2} \left(1 + \frac{c_2}{c_1}\right)$ . The latter is obtained by discarding the non-Gaussian terms,  $n \geq 3$ , in Eq. (16), which is equivalent to approximating the asymptotic distribution of  $\ln T$  by the Gaussian one with the same mean and variance, equal to  $2c_1L$  and  $4c_2L$  respectively. Correspondingly, contribution of the non-Gaussian corrections is represented by the difference  $\Delta\rho_{NG} \equiv \rho - \rho_G = \frac{1}{4} \sum_{n=3}^{\infty} \frac{2^n}{n!} \frac{c_n}{c_1}$ . In Fig. 7 we plot  $\Delta\rho_{NG}$  for the  $\delta$ -correlated and the exponentially correlated disorder (other models of correlation exhibit similar behavior). In agreement with relations (4),  $\Delta\rho_{NG}$  vanishes in the weak disorder limit. Beyond this regime,  $\Delta\rho_{NG}$  is positively peaked at moderate disorder, and the magnitude of the peak increases with  $kR_c$ . Positive values of  $\Delta\rho_{NG}$  mean that the low- $T$  tail of the  $\ln T$ -distribution is heavier than in the Gaussian approximation in a sense that  $\langle T^{-1} \rangle$  and, thus, the relative fluctuation (17), are larger than in the corresponding Gaussian distribution. The result in Fig. 7 shows that this "super-Gaussian" effect is enhanced by the disorder correlations and is most prominent at moderate disorder. Let us stress, however, that this statement applies only to the integral quantity (17) and does not imply any specific form (e.g., super- or sub-Gaussian) for the asymptotic decay law of the low- $T$  tail in the  $\ln T$ -distribution. The latter was studied, e.g., in Refs. [3], [43], [44] using

different methods.

## VI. CONCLUSIONS

We have considered statistical properties of the transmission coefficient  $T$  of a one-dimensional disordered system described by the Schrödinger Eq. (1) with a Gaussian correlated disorder. The main focus of our study was an effect of the correlations on the transmission distribution, which was characterized in terms of the dimensionless ratio between the generalized and the usual Lyapunov exponents,  $\rho = \Lambda/c_1$ , as well as ratios  $c_2/c_1$  and  $c_3/c_1$  of the asymptotic cumulants of  $\ln T^{-1}$ . Both analytical and numerical methods were employed for calculation of  $\Lambda$  and  $c_n$ .

First, a small- $R_c$  approximation was developed for the generalized LE  $\Lambda$ , which is not limited to weak disorder and also is able to account for the non-trivial effects of the correlations. To this end, we obtained an infinite hierarchy of integro-differential equations on the second moments of the wave function and its functional derivatives with respect to the disorder. We have shown that this hierarchical chain can be truncated starting from the second level to obtain a non-trivial approximation, which accounts for the correlations of the disorder. Note that termination of this hierarchy on the first level accounts only for the intensity of the noise, while all information on the correlation properties is lost. This non-trivial closure allowed us to obtain an analytical solution for the generalized LE in an implicit form as a largest real root of a non-linear algebraic equation. The obtained approximation is valid formally for  $kR_c \ll 1$  and  $\Lambda R_c \ll 1$ , where  $k = \sqrt{|\epsilon|}$  is the wave number. It turned out that a standard white noise approximation [36] does not allow to treat the generalized LE in the case of sub-exponentially decaying correlations. To overcome this obstacle, we have proposed a simple self-consistent regularization, which extends the applicability of the approximation to any correlation function satisfying  $|\int_0^\infty C_2(x) dx| < \infty$ . The derived small- $R_c$  approximation was compared to numerical simulations and a good agreement was found.

The asymptotic cumulants of  $\ln T$  were calculated analytically within the semiclassical approximation applicable in a special case of strong disorder,  $\epsilon < V_0$  and  $R_c\sqrt{V_0 - \epsilon} \gg 1$ . For an arbitrary strength of disorder, cumulant coefficients  $c_{1,2,3}$  were obtained by numerical simulations.

Using these analytical and numerical methods, the generalized LE and the first three cumulants of  $\ln T$  were calculated for several models of correlations and for different values of  $R_c$ , which enabled us to investigate effects of correlations on the form of the transmission distribution. In order to study transition between the regimes of weak and strong disorder, we have considered only positive energies  $\epsilon$ , where it was convenient to introduce the dimensionless disorder intensity  $\beta$ , defined in Eq. (49).

In sufficiently weak disorder, we obtained  $c_2/c_1 = 1$ ,

$c_3/c_1 = 0$  and  $\rho = 1$  in all the considered cases. Thus, as expected from previous work (e.g., Refs. [33],[30],[34],[25]), in weak disorder, correlations do not destroy SPS and do not modify the universal relations (4). In the white noise limit,  $kR_c \ll 1$ , this regime is realized for  $c_1/k \sim \beta \ll 1$ . For  $kR_c \gtrsim 1$ , the relevant control parameter is the ratio of the correlation to the localization lengths,  $c_1R_c$ . Quantity  $(c_1R_c)^{-1}$  is interpreted naturally as a measure of randomization of the disorder potential on the scale of the localization length. Correspondingly, the weak disorder universality, expressed by relations (4), takes place when  $c_1R_c \ll 1$  (for  $kR_c \gtrsim 1$ , this means that disorder is weak also in the conventional sense  $c_1/k \ll 1$ ).

Relations (4) are not valid beyond weak disorder. In the white noise model, the corresponding deviation from these relations is weak for any strength of the disorder (for  $\epsilon \gtrsim 0$ ), in a sense that values of  $\rho$  and  $c_2/c_1$  stay very close to unity, while  $c_3/c_1$  remains small compared to unity. On the contrary, in correlated disorder, these quantities depend strongly on the strength of the disorder through the control parameter  $c_1R_c$ . Regarding this parametric dependence, we discuss two regimes:  $c_1R_c \lesssim 1$  and  $c_1R_c \gg 1$ .

The first regime,  $c_1R_c \lesssim 1$ , corresponds to transition from weak to moderate disorder. In this regime and for  $kR_c \gtrsim 1$ , cumulant ratios  $c_2/c_1$  and  $c_3/c_1$  are described by the approximate scaling relations (60) with parameters  $kR_c$  and  $c_1R_c$ . This relations demonstrate that, starting from weak disorder ( $c_1R_c \ll 1$ ), ratios  $c_n/c_1$  increase gradually with parameter  $c_1R_c$  and, for  $c_1R_c \sim 1$ , arrive at values which can be much larger than unity (depending on  $kR_c$ ). In such a case, the bulk of the  $\ln T$ -distribution becomes much broader than in weak or in the white noise disorder.

While the small- $n$  cumulant coefficients  $c_n$  describe bulk of the  $\ln T$ -distribution, LE ratio  $\rho$  is a measure of the extreme relative fluctuation of  $T^{-1}$ , expressed by the ratio between the mean and the typical values of  $T^{-1}$ , Eq. (17). Like the cumulant ratios,  $\rho$  increases from weak to moderate disorder and its deviation from the weak disorder value  $\rho = 1$  is strongly enhanced by the disorder correlations (namely, exceeds unity for  $kR_c \gtrsim 1$ ). As a function of the disorder strength,  $\rho$  and, thus, the extreme relative fluctuation (17), are peaked at moderate disorder (near  $c_1R_c \sim 0.1$ ). This peak of  $\rho$  is associated with the non-Gaussian corrections to the low- $T$  tail of the  $\ln T$ -distribution, whose contribution to  $\rho$  is positively peaked, i.e. "super-Gaussian", at moderate disorder. The latter has the following simple interpretation in terms of the disorder statistics. In moderate disorder, when energy  $\epsilon$  is of the order of  $V_0$ , wave propagation becomes affected by the under-barrier tunneling through the rare but large peaks of the random potential. For  $\sqrt{V_0}R_c \sim kR_c \gtrsim 1$ , already a single large barrier becomes a strong scatterer. Therefore, fluctuations in height and in occurrence of these rare peaks lead to extreme deviations of  $T^{-1}$  from its typical value. In

stronger disorder, the typical value grows significantly, and relative contribution of the large rare barriers becomes less pronounced. This simple explanation can be confirmed by calculating  $\rho$  in the framework of the semiclassical approximation (50) [not presented here].

The second regime,  $c_1R_c \gg 1$ , is realized when  $\epsilon < V_0$  and  $R_c\sqrt{V_0 - \epsilon} \gg 1$ . In these conditions, interference effects are suppressed and localization is dominated by under-barrier tunneling. Then, according to Eq. (50), the transmission distribution is directly related to the statistics of the "excursions" of the random potential above the level  $V(x) = \epsilon$ , and the cumulants of  $\ln T^{-1}$  satisfy the "semiclassical" scaling relations (56), which were verified in numerical simulations. According to Eq. (56), the effect of correlations is expressed by the simple relation  $c_n/c_1 \sim f_n(\epsilon/V_0)(c_1R_c)^{n-1}$ , where the coefficient  $f_n(\epsilon/V_0) \equiv \langle \langle \kappa^n \rangle \rangle / \langle \kappa \rangle^n$  depends only on the one-point distribution of disorder. Semiclassical approximation can also be used to calculate the generalized LE  $\Lambda$ . In particular, for  $|\epsilon| \ll V_0$  and  $\sqrt{V_0}R_c \gg 1$ , we obtain scaling  $\rho \sim (c_1R_c)^{1/3}$ , which is specific to Gaussian statistics of disorder.

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## Appendix A: Exact solution for $\Lambda$ in $\delta$ -correlated disorder

Equation (23) decouples and can be solved exactly for the  $\delta$ -correlated disorder (see e.g. Ref. [3]). Namely, substituting  $C_2(t - \tau) = g\delta(t - \tau)$  into (23) and using the initial condition  $\left\langle \frac{\delta \mathbf{Y}(t^+)}{\delta V(t^-)} \right\rangle = \mathcal{D} \langle \mathbf{Y}(t) \rangle$  (27), one obtains

$$\partial_t \langle \mathbf{Y}(t) \rangle = \mathcal{M}(\epsilon, g) \langle \mathbf{Y}(t) \rangle, \quad (\text{A1})$$

where

$$\mathcal{M}(\epsilon, g) = \mathcal{C} + \frac{g}{2}\mathcal{D}^2 = \sqrt{2} \begin{bmatrix} 0 & 1 & 0 \\ -\epsilon & 0 & 1 \\ g/\sqrt{2} & -\epsilon & 0 \end{bmatrix}, \quad (\text{A2})$$

and matrices  $\mathcal{C}$  and  $\mathcal{D}$  are defined in (21). Solution of Eq. (A1) is

$$\langle \mathbf{Y}(t) \rangle = e^{\mathcal{M}(\epsilon, g)t} \mathbf{Y}_0, \quad (\text{A3})$$

where  $\mathbf{Y}_0 = \mathbf{Y}(0)$  is an initial condition. The generalized LE  $\Lambda$ , Eq. (14), is given by the largest real eigenvalue of the matrix  $\frac{1}{4}\mathcal{M}(\epsilon, g)$ . It is found from the cubic equation

$$\Lambda(4\Lambda^2 + \epsilon) = \frac{g}{8}, \quad (\text{A4})$$

which has the following roots

$$\Lambda_1 = \frac{\rho(g, \epsilon)}{4} - \frac{\epsilon}{3\rho(g, \epsilon)}, \quad \Lambda_{2,3} = \frac{-\Lambda_1 \pm i\sqrt{3\Lambda_1^2 + \epsilon}}{2}, \quad (\text{A5})$$

where  $\rho(g, \epsilon) = \left(g + \sqrt{g^2 + \left(\frac{4}{3}\epsilon\right)^3}\right)^{1/3}$ . The generalized LE  $\Lambda$  is given by  $\Lambda_1$ , which is real and positive. Other two eigenvalues are complex for  $\epsilon > -3\Lambda_1^2$  (i.e.  $\epsilon > -\frac{3}{4}g^{2/3}$ ), and real otherwise. Matrix  $\mathcal{M}(\epsilon, \tilde{g})$  is not normal ( $\mathcal{M}\mathcal{M}^\dagger \neq \mathcal{M}^\dagger\mathcal{M}$ ), and its left and right eigenvectors, corresponding to the eigenvalues  $\Lambda_i$  in Eq. (A5), are

$$u_i^R = \frac{1}{24\Lambda_i^2 + 2\epsilon} \left[ 1, 2\sqrt{2}\Lambda_i, \epsilon + 8\Lambda_i^2 \right]^T, \\ u_i^L = \left[ \epsilon + 8\Lambda_i^2, 2\sqrt{2}\Lambda_i, 1 \right]^\dagger. \quad (\text{A6})$$

These eigenvectors satisfy the normalization  $(u_i^L, u_j^R) = \delta_{ij}$ .

### Appendix B: Born approximation for $\Lambda$

The systematic weak disorder expansion (i.e. in powers of the disorder amplitude  $V_0$ ) for the generalized LE  $\Lambda$  was considered in Ref. [22]. Here we only note that the Born approximation for  $\Lambda$  can be obtained by substitution of the zero order solution for  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$  into (23). The zero order solution for  $\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle$  is obtained by neglecting the noise term in equation (26):

$$\frac{\partial}{\partial t} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle = \mathcal{C} \left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle, \quad \left\langle \frac{\delta \mathbf{Y}(\tau^+)}{\delta V(\tau)} \right\rangle = \mathcal{D} \langle \mathbf{Y}(\tau) \rangle,$$

which has the solution

$$\left\langle \frac{\delta \mathbf{Y}(t)}{\delta V(\tau)} \right\rangle = e^{\mathcal{C}(t-\tau)} \mathcal{D} \langle \mathbf{Y}(\tau) \rangle. \quad (\text{B1})$$

Inserting this solution into Eq. (23) yields an equation closed with respect to  $\langle \mathbf{Y}(t) \rangle$ :

$$\partial_t \langle \mathbf{Y} \rangle = \mathcal{C} \langle \mathbf{Y} \rangle + \int_0^t C_2(t-\tau) \mathcal{D} e^{\mathcal{C}(t-\tau)} \mathcal{D} \langle \mathbf{Y}(\tau) \rangle d\tau, \quad (\text{B2})$$

where pair correlation function  $C_2(t-\tau)$  is defined in Eq. (6). Then, lowest order calculation of the asymptotic growth rate of the solution  $\langle \mathbf{Y}(t) \rangle$  yields the Born approximation

$$\frac{\Lambda}{k} = \frac{g}{8\epsilon^{3/2}} \int_0^\infty ds \Gamma(s) \cos(2kR_c s) + o\left(\frac{g}{8\epsilon^{3/2}}\right), \quad (\text{B3})$$

where  $k = \sqrt{\epsilon}$  and  $g = 2V_0^2 R_c$ . This expression coincides with the Born approximation for LE  $c_1$  [4].

### Appendix C: Method of numerical simulations

The numerical simulations were performed using the tight-binding (TB) model

$$E\psi_n = \varepsilon_n \psi_n - \psi_{n+1} - \psi_{n-1}, \quad (\text{C1})$$

with energy near the band edge,  $0 < E + 2 \ll 1$ , and the diagonal disorder  $|\varepsilon_n| \lesssim E + 2$ . In this regime, Eq. (C1) is a good approximation to the continuous model (5). We relate the continuous model with energy  $\epsilon = k^2$  and potential  $V(x)$  to the TB counterpart by setting

$$E = -2 \cos k, \quad \varepsilon_n = \frac{\sin k}{k} V(x)|_{x=n}. \quad (\text{C2})$$

The correlated Gaussian disorder  $\varepsilon_n$  was generated by filtering sequences of independent Gaussian random variables. The real space convolution with a proper kernel was used to obtain the "short-range" exponential ( $\Gamma(x) = e^{-x}$ ) and Gaussian ( $\Gamma(x) = e^{-\pi x^2/4}$ ) correlations, while the Fourier space filter was applied to obtain the power law correlation  $\Gamma(x) = \text{sinc}^2 \frac{x}{2}$ .

In all simulations presented in this paper, we have fixed the energy at  $E = -1.95$  and varied values of the disorder amplitude  $V_0$  and the correlation radius  $R_c$ . Note that for  $E = -1.95$ , the corresponding wavelength of the solution for a pure system ( $\varepsilon_n = 0$ ) is equal to about 28 sites. Therefore, approximation to a continuum is good as long as  $V_0 \lesssim E$ .

A standard transfer matrix formalism (see e.g. Ref. [45]) was used, in which the TB equation (C1) is rewritten in the matrix form

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \mathcal{T}_n \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad \mathcal{T}_n = \begin{bmatrix} (\varepsilon_n - E) & -1 \\ 1 & 0 \end{bmatrix}, \quad (\text{C3})$$

where  $\mathcal{T}_n$  is a single-site transfer matrix. Then, solution of the initial value problem is given by

$$\begin{pmatrix} \psi_{N+1} \\ \psi_N \end{pmatrix} = \mathcal{T}_{N,1} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \quad \mathcal{T}_{N,1} = \mathcal{T}_N \dots \mathcal{T}_2 \mathcal{T}_1, \quad (\text{C4})$$

where  $\mathcal{T}_{1,N}$  is the total transfer matrix for the system of length  $N$ .

In analogy with Eq. (19), one can introduce vector  $\mathbf{Y}_n = (u_n^2, \sqrt{2}u_nv_n, v_n^2)^T$ , where  $u_n \equiv \psi_n$  and  $v_n \equiv (\psi_n - \psi_{n-1})/k \approx k^{-1}\partial_x \psi$ . Similarly to (C4), the solution for  $\mathbf{Y}_N$  can be write as

$$\mathbf{Y}_{N+1} = \mathcal{K}_{N,1} \mathbf{Y}_1, \quad (\text{C5})$$

where  $\mathbf{Y}_1$  is an initial condition, and  $\mathcal{K}_{N,1}$  is the transfer matrix for  $\mathbf{Y}_N$ , which is readily expressed in terms of the elements of  $\mathcal{T}_{N,1}$ . The largest eigenvalue of  $\mathcal{K}_{N,1}$  is equal to the square modulus of the largest eigenvalue of  $\mathcal{T}_{N,1}$ . Finally, ensemble average over the disorder realizations yields

$$\langle \mathbf{Y}_{N+1} \rangle = \langle \mathcal{K}_{N,1} \rangle \mathbf{Y}_1.$$



Cumulant coefficients  $c_n$ , defined in Eq. (12), are given by the asymptotic linear growth rate of the cumulants  $2^{-n} \langle \ln^n \kappa_N \rangle$  with the system length  $N$ , where  $\kappa_N$  denotes the largest eigenvalue of the matrix  $\mathcal{K}_{1,N}$ . The asymptotic slope was calculated by the linear fit, which have to exclude the region of the initial transient of the order of a few localization lengths. The ensemble average was performed over  $\sim 10^6 \div 10^7$  realizations of disorder.

The generalized Lyapunov exponent  $\Lambda$ , Eq. (14), was calculated as a linear slope of  $\frac{1}{4} \ln \langle \kappa_N \rangle$ . Alternatively,  $\Lambda$  could be found as a slope of the logarithm of the largest eigenvalue of  $\langle \mathcal{K}_{N,1} \rangle$ , which gives practically the same result. About  $10^8$  realization were generated to calculate each value of  $\langle \kappa_N \rangle$ .

Monte-Carlo simulation of the generalized LE can be a quite challenging task, as is briefly explained in the following. In numerical simulation of the generalized LE one have to deal with two restrictions on the system size, both from below and from above. The lower bound is determined by the width of the transient to the asymptotic behavior

$$\ln \langle \kappa_N \rangle = N\Lambda + \text{const.} \quad (\text{C6})$$

Width of this transient is at least of the order of  $R_c$ . This follows from the form of the differential equation (23) for  $\langle Y(t) \rangle$ , which suggests that growth rate of  $\langle Y(t) \rangle$  can not stabilize unless  $t \gg R_c$ . This is because the correlation function in the integral on the right hand side of Eq. (23) decays on the scale of  $R_c$ . In the white noise limit,  $R_c \rightarrow 0$ , the transient region is absent, as follows from the exact analytical result (Appendix A) and was observed numerically. Therefore, we assume that the width of the transient is of the order of  $R_c$ , and other scales, such as the localization length, are less important (unlike the case of  $c_n$ ,  $n > 2$ ).

The upper bound on the system size is determined empirically from the numerical data as a value of  $N$ , beyond which the linear dependence in Eq. (C6) is violated. This computational artifact originates from the insufficient statistics in averaging of the broadly distributed quantity  $\kappa_N$ , which is equivalent to  $T^{-1}$ . According to Eqs. (2) and (4), distribution of  $\kappa_N$  is log-normal in weak disorder, while some corrections to the limiting log-normal form appear in stronger disorder. As follows from Eqs. (13) and (17), this distribution becomes increasingly broad and heavy-tailed with the increase of the dimensionless system length  $l = c_1 N$ . For large  $l$ , long tails of the distribution, which dominate the theoretical mean of  $\kappa_N$ , are typically under-sampled in simulations with a finite number of realization. As a result, the obtained values of  $\ln \langle \kappa_N \rangle$  become typically underestimated (formally, expectation value of  $\ln \langle \kappa_N \rangle$  becomes smaller than  $\ln \langle \kappa_N \rangle_\infty$ , where  $\langle \kappa_N \rangle_\infty$  is the theoretical mean). This effect increases with the system size, the relevant length scale being the localization length  $c_1^{-1}$ . Therefore, the upper bound on the system length is of the order of a few  $c_1^{-1}$ . Thus, the upper and the lower bounds even-

tually coincide in sufficiently strong disorder, since  $c_1^{-1}$  becomes small. In such a case, numerical calculation of  $\Lambda$  becomes impossible, unless the number of the realizations is increased dramatically (for the exactly log-normal  $\kappa_N$ , it can be shown that the improvement is logarithmically slow). In moderate disorder, the small range between the lower and the upper bounds results in large uncertainty in the calculated  $\Lambda$ , as indicated by the error bars in Figs. 1 and 2.

#### Appendix D: Auxiliary formulas for semiclassical approximation

Cumulants  $\langle \langle \kappa^n \rangle \rangle$  are easily written in terms of  $\langle \langle \kappa^n \rangle \rangle_\epsilon$  and  $\langle \langle \Theta_\epsilon^n(V) \rangle \rangle$  using that  $\Theta_\epsilon^n(V) = \Theta_\epsilon(V)$  and  $\langle \kappa^n \rangle = \langle \kappa^n \rangle_\epsilon \langle \Theta_\epsilon(V) \rangle$ , where  $\langle \dots \rangle_\epsilon$  stands for the average with the distribution  $P_\epsilon(V) \equiv \Theta_\epsilon(V) P(V) / \int_\epsilon^\infty dV P(V)$ , introduced in Sec. IV. For example, the second cumulant,  $\langle \langle \kappa^2 \rangle \rangle \equiv \langle \kappa^2 \rangle - \langle \kappa \rangle^2$ , becomes

$$\langle \langle \kappa^2 \rangle \rangle = \langle \kappa^2 \rangle_\epsilon \langle \Theta_\epsilon(V) \rangle - \langle \kappa \rangle_\epsilon^2 \langle \Theta_\epsilon(V) \rangle^2, \quad (\text{D1})$$

which, using  $\langle \langle \Theta_\epsilon^2(V) \rangle \rangle = \langle \Theta_\epsilon(V) \rangle - \langle \Theta_\epsilon(V) \rangle^2$  and  $\langle \langle \kappa^2 \rangle \rangle_\epsilon = \langle \kappa^2 \rangle_\epsilon - \langle \kappa \rangle_\epsilon^2$ , yields after some rearrangement

$$\langle \langle \kappa^2 \rangle \rangle = \langle \langle \kappa^2 \rangle \rangle_\epsilon \langle \Theta_\epsilon(V) \rangle + \langle \kappa \rangle_\epsilon^2 \langle \langle \Theta_\epsilon^2(V) \rangle \rangle. \quad (\text{D2})$$

Similar expansion for the third cumulant,  $\langle \langle \kappa^3 \rangle \rangle = \langle \kappa^3 \rangle - 3 \langle \kappa^2 \rangle \langle \kappa \rangle + 2 \langle \kappa \rangle^3$ , gives expression (57) for  $f_3 = \langle \langle \kappa^3 \rangle \rangle / \langle \kappa \rangle^3$ .

For Gaussian distribution  $P(V) = (2\pi V_0^2)^{-1/2} e^{-V^2/2V_0^2}$ , definition (51) yields the following expression for LE:

$$c_1 = V_0^{1/2} 2^{-9/4} e^{-r^2/2} U\left(\frac{3}{4}, \frac{1}{2}, \frac{r^2}{2}\right), \quad r \equiv \frac{\epsilon}{V_0}, \quad (\text{D3})$$

while functions  $f_n\left(\frac{\epsilon}{V_0}\right) \equiv \langle \langle \kappa^n \rangle \rangle / \langle \kappa \rangle^n$  for  $n = 2, 3$  are given by ( $r \geq 0$ )

$$\begin{aligned} f_2(r) &= -1 + 8 \frac{2e^{r^2/2} - \sqrt{2\pi} r e^{r^2} \text{erfc}(r/\sqrt{2})}{\sqrt{\pi} U^2\left(\frac{3}{4}, \frac{1}{2}, \frac{r^2}{2}\right)}, \\ f_3(r) &= 2 - 24 \frac{2e^{\frac{r^2}{2}} - \sqrt{2\pi} r e^{r^2} \text{erfc}\frac{r}{\sqrt{2}}}{\sqrt{\pi} U^2\left(\frac{3}{4}, \frac{1}{2}, \frac{r^2}{2}\right)} + \\ &\quad + \frac{32r^{\frac{1}{2}} e^{\frac{5r^2}{4}} \left[ (1+r^2) K_{\frac{1}{4}}\left(\frac{r^2}{4}\right) - r^2 K_{\frac{3}{4}}\left(\frac{r^2}{4}\right) \right]}{2^{1/4} \sqrt{\pi} U^3\left(\frac{3}{4}, \frac{1}{2}, \frac{r^2}{2}\right)} \end{aligned} \quad (\text{D4})$$

where  $U(a, b, z)$  is a confluent hypergeometric function of the second kind (or Tricomi function) and  $K_\nu(z)$  is a modified Bessel function of the second kind [46].

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- [1] P. Hänggi and P. Jung, *Adv. Chem. Phys.* **89**, 239 (1995).
- [2] P.W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
- [3] R. Zillmer and A. Pikovsky, *Phys. Rev. E* **67**, 061117 (2003).
- [4] I.M. Lifshitz, S.A. Gredeskul, and L.A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
- [5] H. Schomerus and M. Titov, *Phys. Rev. E* **66**, 066207 (2002).
- [6] G. Paladin and A. Vulpiani, *Phys. Rev. B* **35**, 2015 (1987).
- [7] S. Kotani, *Proceedings of Conference on Stochastic Analysis Kyoto*, (1982).
- [8] B. Simon, *Commun. Math. Phys.* **89**, 227 (1983).
- [9] S. Kotani and B. Simon, *Commun. Math. Phys.* **112**, 103 (1987).
- [10] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag, Berlin (1992).
- [11] R. Landauer, *Philos. Mag.* **21**, 863 (1970).
- [12] D. S. Wiersma *et al.*, *Nature* **390**, 671673 (1997); T. Schwartz *et al.*, *Nature* **446**, 5255 (2007); Y. Lahini *et al.*, *Phys. Rev. Lett.* **100**, 013906 (2008).
- [13] A. A. Chabanov, M. Stoytchev, and A. Z. Genack, *Nature (London)* **404**, 850 (2000); U. Kuhl, F.M. Izrailev, A.A. Krohin, and H.-J. Stockmann, *Appl. Phys. Lett.* **77**, 633 (2000).
- [14] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, *Nature (London)* **453**, 891 (2008); G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, *Nature* **453**, 895 (2008).
- [15] H. Hu, A. Strybulevych, J. H. Page, S. E. Skipetrov, and B. A. van Tiggelen, *Nature Physics* **4**, 945 (2008).
- [16] T. Ernst, T. Paul, P. Schlagheck, *Phys. Rev. A* **81**, 013631 (2010).
- [17] D. Clément, A.F. Varon, J.A. Retter, L. Sanchez-Palencia, A. Aspect and P. Bouyer, *New J. Phys.* **8**, 165 (2006).
- [18] F.M. Izrailev and A.A. Krokhin, *Phys. Rev. Lett.* **82**, 4062 (1999).
- [19] L. Tessieri and F. M. Izrailev, *Phys. Rev. E* **64**, 66120 (2001).
- [20] E. Gurevich and O. Kenneth, *Phys. Rev. A* **79**, 063617 (2009).
- [21] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Grémaud, C.A. Müller, and C. Miniatura, *Phys. Rev. A*, **80**, 023605 (2009).
- [22] L. Tessieri, *J. Phys. A: Math. Gen.* **35**, 9585 (2002).
- [23] A. Iomin, *Phys. Rev. E* **79**, 062102 (2009).
- [24] M. Titov and H. Schomerus, *Phys. Rev. Lett.* **95**, 126602 (2005).
- [25] L. I. Deych, M. V. Eremenchouk, and A. A. Lisyansky, *Phys. Rev. B* **67**, 024205 (2003).
- [26] D. H. Dunlap, H. L. Wu, and P. W. Phillips, *Phys. Rev. Lett.* **65**, 88 (1990); F. A. B. F. de Moura and M. L. Lyra, *Phys. Rev. Lett.* **81**, 3735 (1998); T. Kaya, *Eur. Phys. J. B* **55**, 49 (2007); A.M. Garcia-Garcia and E. Cuevas, *Phys. Rev. B* **79**, 073104 (2009).
- [27] P. W. Anderson, D.J. Thouless, E. Abrahams and D.S. Fisher, *Phys. Rev. B* **22**, 3519 (1980).
- [28] A. Cohen, Y. Roth and B. Shapiro, *Phys. Rev. B* **38**, 12125 (1988).
- [29] B. Shapiro, *Philos. Mag. B* **56**, 1031 (1987).
- [30] M. J. de Oliveira and A. Petri, *Phys. Rev. E* **53**, 2960 (1996); *Phys. Rev. B* **56**, 251 (1997).
- [31] A.A. Abrikosov, *Solid State Commun.* **37**, 997 (1981).
- [32] L. I. Deych, A. A. Lisyansky, and B. L. Altshuler, *Phys. Rev. Lett.* **84**, 2678 (2000); *Phys. Rev. B* **64**, 224202 (2001).
- [33] N. Nishiguchi, S. I. Tamura, and F. Nori, *Phys. Rev. B* **48**, 2515 (1993); **48**, 14426 (1993).
- [34] L. I. Deych, D. Zaslavsky, and A. A. Lisyansky, *Phys. Rev. E* **56**, 4780 (1997).
- [35] H. Schomerus and M. Titov, *Phys. Rev. B* **67**, 100201(R) (2003).
- [36] V.I. Kliatskin, *Stochastic equations and waves in randomly inhomogeneous media* (Nauka, Moskva, 1980) (in Russian).
- [37] P.A. Mello and N. Kumar, *Quantum transport in mesoscopic systems: complexity and statistical fluctuations*, (Oxford University Press, Oxford, 2004).
- [38] Correlation between the transmission coefficient and the reflection and transmission phases is not exactly zero, and it can not be ignored in certain circumstances. One such example is calculation of the transmission (conductance) distribution based on the phase formalism [35]. For our purposes, however, this correlation becomes negligible in the limit of long system.
- [39] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, New York, 1985).
- [40] J.W. Goodman, "Statistical properties of Laser Speckle Patterns" in "Laser Speckle and Related Phenomena", edited by J.C. Dainty, 2nd Ed., Springer-Verlag 1984.
- [41] D. Blacknell, M. Jahangir, F. Piccoloand, R. J. A. Tough, *J. Phys. A: Math. Gen.* **34**, 1231 (2001).
- [42] According to our semiclassical treatment of strong disorder (Sec. IV, Eq. (59)),  $\rho$  scales as  $(\sqrt{V_0}R_c)^{1/3}$  for  $|\epsilon| \ll V_0$  and  $\sqrt{V_0}R_c \gg 1$ . Thus, if  $V_0$  is increased with  $\epsilon$  and  $R_c$  fixed, then the local peak of  $\rho$  at moderate disorder is followed by subsequent increase at much stronger disorder. The latter is not present in Fig. 3, which corresponds to less strong disorder,  $\sqrt{V_0}R_c \lesssim 1$ . In fact, we checked that the non-monotonic dependence of  $\rho$  on  $V_0$  in moderate and strong disorder can be obtained using the semiclassical approximation (50).
- [43] I.E. Smolyarenko, B.L. Altshuler, *Phys. Rev. B* **55**, 10451 (1997).
- [44] V.M. Apalkov, M.E. Raikh, *Semiconductors* **42**, 940 (2008).
- [45] Y. Liu and K.A. Chao, *Phys. Rev. B* **34**, 5247 (1986).
- [46] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1972).